Separation of $\mathsf{MA}^+(\sigma\text{-closed})$ from stationary reflection principles

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Abstract

$\mathsf{MA}^+(\sigma\text{-closed})$ was introduced by Foreman-Magidor-Shelah [5], and they proved that $\mathsf{MA}^+(\sigma\text{-closed})$ implies the stationary reflection principle in $[\lambda]^\omega$ for all $\lambda \geq \omega_2$. On the contrary we prove that $\mathsf{MA}^+(\sigma\text{-closed})$ does not follow even from stronger stationary reflection principles. More precisely we show that $\mathsf{MA}^+(\sigma\text{-closed})$ does not follow from a slightly strengthening of the Projectively Stationary Reflection Principle and the Diagonal Reflection Principle to Internally Approachable Sets.

1 Introduction

$\mathsf{MA}^+(\mathbb{P})$ is the following strengthening of the forcing axiom for a poset $\mathbb{P}$:

> For any family $\mathcal{A}$ of ω_1-many maximal antichains of $\mathbb{P}$ and any $\mathbb{P}$-name $\dot{S}$ for a stationary subset of ω_1 there is a filter $g \subseteq \mathbb{P}$ such that g is $\mathcal{A}$-generic, i.e. $g \cap A \neq \emptyset$ for all $A \in \mathcal{A}$, and such that the set
>
> $$\dot{S}_g := \{\alpha < \omega_1 \mid \exists p \in g,\ p \Vdash \text{“}\alpha \in \dot{S}\text{”}\}$$
>
> is stationary in ω_1.

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$\mathsf{MA}^+(\sigma\text{-closed})$ is the assertion that $\mathsf{MA}^+(\mathbb{P})$ holds for all σ-closed posets $\mathbb{P}$. This was first introduced by Foreman-Magidor-Shelah [5].

It was proved in [5] that $\mathsf{MA}^+(\sigma\text{-closed})$ implies the stationary reflection principle in $[\lambda]^\omega$ for all $\lambda \geq \omega_2$, which is often called the Weak Reflection Principle. In fact it implies the stationary reflection principle to internally approachable sets of length ω_1, which we denote as $\mathsf{RP}(\mathrm{IA})$. To present a precise statement of $\mathsf{RP}(\mathrm{IA})$, first we give some notation. A sequence $\langle M_\alpha \mid \alpha < \gamma\rangle$, $\gamma \in \mathrm{On}$, is called a *nice* $\in$*-chain* if $\langle M_\alpha \mid \alpha \leq \beta\rangle \in M_{\beta+1}$ for all β with $\beta+1 < \gamma$. A set N is said to be *internally approachable of length* γ if there is a nice $\in$-chain $\langle M_\alpha \mid \alpha < \gamma\rangle$ such that $\bigcup_{\alpha<\gamma} M_\alpha = N$. Then $\mathsf{RP}(\mathrm{IA})$ is the following assertion:

> For any regular $\lambda \geq \omega_2$ and any stationary $X \subseteq [\mathcal{H}_\lambda]^\omega$ there is an internally approachable $N \in [\mathcal{H}_\lambda]^{\omega_1}$ of length ω_1 such that $X \cap [N]^\omega$ is stationary.

It was proved by Krueger [9] that $\mathsf{RP}(\mathrm{IA})$ is strictly stronger than the Weak Reflection Principle.

Recently Miyamoto formulated $\mathsf{MA}^*(\sigma\text{-closed})$ and $\mathsf{MA}'(\sigma\text{-closed})$, which are forcing axioms similar to each other, and Usuba [11] proved that they are equivalent to $\mathsf{MA}^+(\sigma\text{-closed})$ and $\mathsf{RP}(\mathrm{IA})$, respectively. Then it is natural to ask whether $\mathsf{MA}^+(\sigma\text{-closed})$ and $\mathsf{RP}(\mathrm{IA})$ are equivalent. In this paper we prove that $\mathsf{MA}^+(\sigma\text{-closed})$ is strictly stronger than $\mathsf{RP}(\mathrm{IA})$. In fact we show that $\mathsf{MA}^+(\sigma\text{-closed})$ follows from neither of two strengthenings of $\mathsf{RP}(\mathrm{IA})$ below:

The first one is a variant of the Projectively Stationary Reflection Principle introduced by Feng-Jech [4], which will be denoted as $\mathsf{PSR}(\mathrm{IA})$. First recall the notion of projectively stationary sets: For $W \supseteq \omega_1$ and $Z \subseteq [W]^\omega$, we say that Z is *projectively stationary* in $[W]^\omega$ if the set $\{x \in Z \mid x \cap \omega_1 \in S\}$ is stationary in $[W]^\omega$ for any stationary $S \subseteq \omega_1$. Then $\mathsf{PSR}(\mathrm{IA})$ is the following assertion:

> For any regular cardinal $\lambda \geq \omega_2$ and any projectively stationary $Z \subseteq [\mathcal{H}_\lambda]^\omega$ there is a $\subseteq$-increasing continuous nice $\in$-chain $\langle M_\alpha \mid \alpha < \omega_1\rangle$ through Z.

The original Projectively Stationary Reflection Principle, PSR, is the assertion obtained by omitting "nice" in $\mathsf{PSR}(\mathrm{IA})$. In [4] it was proved that MM implies $\mathsf{PSR}(\mathrm{IA})$ and that PSR is equivalent to the Strong Reflection Principle by Todorčević [2] [1]. It was also shown in [4] that PSR implies the Weak Reflection Principle, and the same argument shows that $\mathsf{PSR}(\mathrm{IA})$ implies $\mathsf{RP}(\mathrm{IA})$.

[1] Several variations of the Strong Reflection Principle have been produced as many set theorists dealt with them. But Todorčević told the author that the original version is the one equivalent to $\mathsf{PSR}(\mathrm{IA})$.

The second strengthening of $\mathsf{RP}(\mathrm{IA})$ is the Diagonal Reflection Principle to Internally Approachable Sets, $\mathsf{DRP}(\mathrm{IA})$, introduced by Cox [3]. As far as the author knows, this is the strongest one among the simultaneous refleciton principles for stationary sets consisting of countable sets. $\mathsf{DRP}(\mathrm{IA})$ is the following statement:

> For any cardinal $\lambda \geq \omega_2$ and any regular cardinal $\bar{\lambda} > \lambda$ there are stationary many $N \in [\mathcal{H}_{\bar{\lambda}}]^{\omega_1}$ such that N is internally approachable of length ω_1 and such that $X \cap [N \cap \mathcal{H}_\lambda]^\omega$ is stationary for all stationary $X \subseteq [\mathcal{H}_\lambda]^\omega$ with $X \in N$.

It is easy to see that $\mathsf{DRP}(\mathrm{IA})$ implies $\mathsf{RP}(\mathrm{IA})$. It is also proved in [3] that $\mathsf{MA}^{++}(\sigma\text{-closed})$ implies $\mathsf{DRP}(\mathrm{IA})$.

As we mentioned before, we will prove that neither $\mathsf{PSR}(\mathrm{IA})$ nor $\mathsf{DRP}(\mathrm{IA})$ implies $\mathsf{MA}^{+}(\sigma\text{-closed})$. In fact, it will be proved that $\mathsf{PSR}(\mathrm{IA}) + \mathsf{DRP}(\mathrm{IA})$ does not imply $\mathsf{MA}^{+}(\sigma\text{-closed})$:

Theorem 1. *Assume there is a supercompact cardinal. Then there is a forcing extension in which both* $\mathsf{PSR}(\mathrm{IA})$ *and* $\mathsf{DRP}(\mathrm{IA})$ *hold, but* $\mathsf{MA}^{+}(\sigma\text{-closed})$ *fails.*

Here note that $\mathsf{DRP}(\mathrm{IA})$ is consistent with CH, while $\mathsf{PSR}(\mathrm{IA})$ implies $2^\omega = \omega_2$. (See [3] and [2].) We will also prove the following:

Theorem 2. *Assume there is a supercompact cardinal. Then there is a forcing extension in which both* $\mathsf{DRP}(\mathrm{IA})$ *and* CH *hold, but* $\mathsf{MA}^{+}(\sigma\text{-closed})$ *fails.*

This paper is constructed as follows: In Section 2 we will give our notation and basic facts used in this paper. In Section 3 we present a combinatorial principle Φ which denies $\mathsf{MA}^{+}(\sigma\text{-closed})$. There we also present a forcing notion relevant to Φ. In Section 4 we will prove the consistency of $\mathsf{PSR}(\mathrm{IA}) + \mathsf{DRP}(\mathrm{IA}) + \Phi$, which implies Theorem 1. In Section 5 we will prove the consistency of $\mathsf{DRP}(\mathrm{IA}) + \mathsf{CH} + \Phi$, which implies Theorem 2.

2 Preliminaries

In this section we present our notation and basic facts used in this paper. Consult Jech [6] and Kanamori [7] for those which are not presented here.

We begin with our notation relevant to trees. Let κ be an ordinal. For $u, v \in {}^{<\kappa}2$ we say that u and v are *comparable* if either $u \subseteq v$, or $v \subseteq u$. For $T \subseteq {}^{<\kappa}2$, a κ-*branch* of T is $B \in {}^{\kappa}2$ such that the set $\{\alpha < \kappa \mid B \restriction \alpha \in T\}$ is cofinal in κ.

Next we give our notation on stationary sets. Let κ be a cardinal and W be a set including κ. $Z \subseteq [W]^\kappa$ is said to be *club* in $[W]^\kappa$ if Z is $\subseteq$-cofinal in $[W]^\kappa$, and $\bigcup_{\alpha<\beta} z_\alpha \in Z$ for any $\beta \leq \kappa$ and any $\subseteq$-increasing sequence $\langle z_\alpha \mid \alpha < \beta\rangle$ in Z. $X \subseteq [W]^\kappa$ is said to be *stationary* in $[W]^\kappa$ if $X \cap Z \neq \emptyset$ for all club $Z \subseteq [W]^\kappa$. It is well-known that X is stationary in $[W]^\kappa$ if and only if for any function $f : [W]^{<\omega} \to W$ there exists $x \in X$ such that x is closed under f and such that $\kappa \subseteq x$. Moreover the condition that $\kappa \subseteq x$ can be omitted if $\kappa = \omega$.

The rest of this section is devoted to our notation and basic facts relevant to the forcing.

Let $\mathbb{P}$ be a forcing notion. Then $1_\mathbb{P}$ denotes the largest element of $\mathbb{P}$. If we say "$\cdots$ in $V^\mathbb{P}$", then it means that $1_\mathbb{P}$ forces $\cdots$. Similarly, if we say "$\dot{a}$ is a $\mathbb{P}$-name for $\cdots$", then it means that $1_\mathbb{P}$ forces that $\dot{a}$ is $\cdots$. If definable objects such as ω_1 and $\mathcal{H}_\theta$ appear in $\cdots$, then they denote those defined in $V^\mathbb{P}$.

Let $\mathbb{P}$ be a poset and κ be a cardinal. We say that $\mathbb{P}$ is $<\kappa$-*Baire* if for any family $\mathcal{A}$ of $<\kappa$-many maximal antichains in $\mathbb{P}$ there are densely many $p \in \mathbb{P}$ which meets all elements of $\mathbb{P}$. Note that $\mathbb{P}$ is $<\kappa$-*Baire* if and only if any forcing extension by $\mathbb{P}$ does not add any new sequences of ordinals of length $<\kappa$. We say that $\mathbb{P}$ is σ-Baire if it is $<\omega_1$-Baire.

Let κ be a regular cardinal. Then $\mathrm{Add}(\kappa)$ denotes the poset ${}^{<\kappa}2$ ordered by reverse inclusions. For a set W let $\mathrm{Col}(\kappa, W)$ be the poset ${}^{<\kappa}W$ ordered by reverse inclusions. For ordinals ν and $\lambda > \nu$, $\mathrm{Col}(\kappa, \nu, <\lambda)$ denotes the $<\kappa$-support product of $\langle \mathrm{Col}(\kappa, \alpha) \mid \nu \leq \alpha < \lambda\rangle$. $\mathrm{Col}(\kappa, 0, <\lambda)$ will be simply denoted as $\mathrm{Col}(\kappa, <\lambda)$. Note that all $\mathrm{Add}(\kappa)$, $\mathrm{Col}(\kappa, W)$ and $\mathrm{Col}(\kappa, \nu, <\lambda)$ are $<\kappa$-closed.

Let M be a set and $\mathbb{P}$ be a poset. An $(M, \mathbb{P})$-*generic sequence* is a descending sequence $\langle p_\xi \mid \xi < \zeta\rangle$ in $\mathbb{P} \cap M$ for some ordinal ζ such that any maximal antichain $A \in M$ in $\mathbb{P}$ is met by some p_ξ. We say that $p \in \mathbb{P}$ is $(M, \mathbb{P})$-*generic* if $A \cap M$ is predense below p for any maximal antichain $A \in M$ in $\mathbb{P}$. Moreover $p \in \mathbb{P}$ is said to be *strongly* $(M, \mathbb{P})$-*generic* if p meets $A \cap M$ for every maximal antichain $A \in M$ in $\mathbb{P}$. A filter $g \subseteq \mathbb{P} \cap M$ is said to be $(M, \mathbb{P})$-*generic* if $g \cap A \cap M \neq \emptyset$ for any maximal antichain $A \in M$ in $\mathbb{P}$. Furthermore, for an $\mathbb{P}$-generic filter G over V let

$$M[G] \;:=\; \{\dot{a}_G \mid \dot{a} \text{ is a } \mathbb{P}\text{-name} \wedge\ \dot{a} \in M\}\,,$$

where $\dot{a}_G$ denotes an interpretation of $\dot{a}$ by G.

Let $\mathbb{P}$ be a poset. Then $\mathsf{MA}(\mathbb{P})$ is the assertion that for any family $\mathcal{A}$ of ω_1-many maximal antichains in $\mathbb{P}$ there is an $\mathcal{A}$-generic filter $g \subseteq \mathbb{P}$. Moreover $\mathsf{MA}^{++}(\mathbb{P})$ is the further strengthening of $\mathsf{MA}^{+}(\mathbb{P})$ stated as follows:

For any family $\mathcal{A}$ of ω_1-many maximal antichains in $\mathbb{P}$ and any family $\mathcal{S}$ of ω_1-many $\mathbb{P}$-names for stationary subsets of ω_1, there exists a $\mathcal{A}$-generic filter $g \subseteq \mathbb{P}$ such that $\dot{S}_g$ is stationary for every $\dot{S} \in \mathcal{S}$.

Let PFA^+ and PFA^{++} be the +-version and ++-version of the Proper Forcing Axiom PFA. Similarly, MM^+ and MM^{++} are the +-version and ++-version of Martin's Maximum MM. In [5] it is proved that if there is a supercompact cardinal, then there exists a forcing extension in which MM^+ holds. In fact it is easy to see that MM^{++} holds in that model. The following characterizations of $\mathsf{MA}(\mathbb{P})$ and $\mathsf{MA}^{++}(\mathbb{P})$, due to Woodin [12], are well-known:

Fact 2.1 (Woodin [12])**.** *Let $\mathbb{P}$ be a poset.*

(1) $\mathsf{MA}(\mathbb{P})$ *holds if and only if for any suffciently large regular cardinal θ there are stationary many $M \in [\mathcal{H}_\theta]^{\omega_1}$ such that an $(M, \mathbb{P})$-generic filter exists.*

(2) $\mathsf{MA}^{++}(\mathbb{P})$ *holds if and only if for any sufficiently large regular cardinal θ there are stationary many $M \in [\mathcal{H}_\theta]^{\omega_1}$ such that there is an $(M, \mathbb{P})$-generic filter g with $\dot{S}_g$ stationary for every $\mathbb{P}$-name $\dot{S} \in M$ of a stationary subset of ω_1.*

3 Φ and F-iteration

In this section we present a combinatorial principle Φ denying $\mathsf{MA}^+(\mathrm{Add}(\omega_1))$. As we mentioned at the introduction, we will prove that Φ is consistent with reflection principles in later sections. Here we also give a relevant forcing notion $\mathbb{Q}(F)$ in Subsection 3.2, which will be used in these consistency proofs.

3.1 Φ and $\mathsf{MA}^+(\mathrm{Add}(\omega_1))$

Here we present Φ and prove that it denies $\mathsf{MA}^+(\mathrm{Add}(\omega_1))$.

First we prepare notation: We call $e : {}^{<\omega}2 \to {}^{<\omega_1}2$ a $\subseteq$*-preserving map* if $e(s) \subseteq e(t)$ for any $s, t \in {}^{<\omega}2$ with $s \subseteq t$. If $e : {}^{<\omega}2 \to {}^{<\omega_1}2$ is $\subseteq$-preserving, then let $e^\circ : {}^{\omega}2 \to {}^{<\omega_1}2$ be the function such that $e^\circ(b) = \bigcup_{n<\omega} e(b \restriction n)$.

Now let Φ be the following assertion:

There exists a function $F : {}^{<\omega_1}2 \to 2$ satisfying [I] and [II] below:

[I] For any $\subseteq$-preserving $e : {}^{<\omega}2 \to {}^{<\omega_1}2$ with e° injective, there is $b \in {}^{\omega}2$ with $F(e^\circ(b)) = 1$.

[II] For any $B \in {}^{\omega_1}2$ the set $\{\alpha < \omega_1 \mid F(B \restriction \alpha) = 1\}$ is non-stationary.

Lemma 3.1. *Φ denies $\mathsf{MA}^+(\mathrm{Add}(\omega_1))$. Hence Φ denies $\mathsf{MA}^+(\sigma\text{-closed})$.*

Proof. Assume Φ, and let F be a witness of Φ. Let $\dot{G}$ be the canonical name for an $\mathrm{Add}(\omega_1)$-generic filter, and let $\dot{S}$ be an $\mathrm{Add}(\omega_1)$-name for the set of all $\alpha < \omega_1$ with $F((\bigcup \dot{G}) \restriction \alpha) = 1$. By [II], $\dot{S}_g$ is non-stationary for any $\{{}^{\alpha}2 \mid \alpha < \omega_1\}$-generic filter g. Thus it suffices to show that $\dot{S}$ is stationary in $V^{\mathrm{Add}(\omega_1)}$.

Take an arbitrary $u \in \mathrm{Add}(\omega_1)$ and an arbitrary $\mathrm{Add}(\omega_1)$-name $\dot{C}$ of a club subset of ω_1. We will find $u^* \leq u$ and $\alpha < \omega_1$ such that $u^* \Vdash$ "$\alpha \in \dot{S} \cap \dot{C}$". Take a sufficiently large regular cardinal θ and a countable $M \prec \langle \mathcal{H}_\theta, \in, u, \dot{C} \rangle$. Let $\alpha := M \cap \omega_1$, and let $\langle A_n \mid n < \omega \rangle$ be an enumeration of all maximal antichains in $\mathrm{Add}(\omega_1)$ which belong to M. Then for each $s \in {}^{<\omega}2$ we can inductively take $e(s) \in \mathrm{Add}(\omega_1) \cap M$ so that

(i) $u \subseteq e(\emptyset)$, and $e(s) \subseteq e(t)$ if $s \subseteq t$,

(ii) $e(s)$ and $e(t)$ are incomparable for any incomparable s and t,

(iii) $e(s)$ meets $A_{\mathrm{dom}(s)}$.

Then e is $\subseteq$-preserving, and e° is injective. By [I] take $b \in {}^{\omega}2$ with $F(e^\circ(b)) = 1$, and let $u^* := e^\circ(b)$. Note that $\mathrm{dom}(u^*) = \alpha$. Then u^* forces that $F((\bigcup \dot{G}) \restriction \alpha) = F(u^*) = 1$, that is, $\alpha \in \dot{S}$. Moreover u^* forces that $\alpha \in \dot{C}$ because u^* is strongly $(M, \mathrm{Add}(\omega_1))$-generic. □

3.2 F-iteration

In this subsection we present a forcing notion $\mathbb{Q}(F)$ which forces a given function $F : {}^{<\omega_1}2 \to 2$ to witness Φ.

First we introduce an F-iteration, which is essentially a countable support iteration of club shootings. For an ordinal ξ let

Γ_ξ := the set of all countable partial functions q on ξ such that $q(\eta)$ is a closed bounded subset of ω_1 for each $\eta \in \mathrm{dom}(q)$.

For a function $F : {}^{<\omega_1}2 \to 2$ and an ordinal ζ we call $\langle \mathbb{Q}_\xi, \dot{B}_\eta \mid \xi \leq \zeta, \eta < \zeta \rangle$ an *F-iteration of length ζ* if the following hold for all $\xi \leq \zeta$ and $\eta < \zeta$:

- $\mathbb{Q}_\xi$ is a poset such that $\mathbb{Q}_\xi \subseteq \Gamma_\xi$ and such that $q_0 \leq q_1$ in $\mathbb{Q}_\xi$ if and only if $\mathrm{dom}(q_0) \supseteq \mathrm{dom}(q_1)$, and $q_0(\eta)$ is an end-extension of $q_1(\eta)$ for each $\eta \in \mathrm{dom}(q_1)$.

- $\dot{B}_\eta$ is a nice $\mathbb{Q}_\eta$-name for an $(\omega_1)^V$-branch of $({}^{<\omega_1}2)^V$.
- $\mathbb{Q}_{\eta+1}$ consists of all $q \in \Gamma_{\eta+1}$ such that $q \restriction \eta \in \mathbb{Q}_\eta$ and such that if $\eta \in \mathrm{dom}(q)$, then $q \restriction \eta \Vdash$ "$F(\dot{B}_\eta \restriction \alpha) = 0$" for all limit point α of $q(\eta)$.
- If ξ is a limit ordinal, then $\mathbb{Q}_\xi$ consists of all $q \in \Gamma_\xi$ such that $q \restriction \xi' \in \mathbb{Q}_{\xi'}$ for all $\xi' < \xi$.

If $\langle \mathbb{Q}_\xi, \dot{B}_\eta \mid \xi \leq \zeta, \eta < \zeta \rangle$ is an F-iteration, then $\mathbb{Q}_\zeta$ is also called an F-iteration.

The standard argument using the Δ-system lemma proves the following:

Lemma 3.2. *For any $F : {}^{<\omega_1}2 \to 2$ every F-iteration has the $(2^\omega)^+$-c.c.*

Let $\mu := 2^{2^\omega}$. By Lemma 3.2, using the book-keeping method, we can construct an F-iteration $\langle \mathbb{Q}_\xi, \dot{B}_\eta \mid \xi \leq \mu, \eta < \mu \rangle$ such that every nice $\mathbb{Q}_\mu$-name for an $(\omega_1)^V$-branch of $({}^{<\omega_1}2)^V$ appears among $\{\dot{B}_\eta \mid \eta < \mu\}$. Let $\mathbb{Q}(F)$ be such $\mathbb{Q}_\mu$. The following is easily checked:

Lemma 3.3. *Suppose that $F : {}^{<\omega_1}2 \to 2$ and that $\mathbb{Q}(F)$ is σ-Baire. Then F satisfies [II] in $V^{\mathbb{Q}(F)}$. If F satisfies [I] in V in addition, then F also satisfies [I] in $V^{\mathbb{Q}(F)}$, and so F witnesses Φ in $V^{\mathbb{Q}(F)}$.*

But $\mathbb{Q}(F)$ is not σ-Baire in general. For example, if $F(u) = 1$ for all $u \in {}^{<\omega_1}2$, then $\mathbb{Q}(F)$ clearly collapses ω_1. Below we give conditions of F which imply that F-iterations are σ-Baire. We also give conditions of F assuring that F-iterations preserve a given stationary set, which will be a key for our proof of Theorem 1 and 2.

First let [III] be the following condition for a function $F : {}^{<\omega_1}2 \to 2$:

[III] For any countable family $\mathcal{E}$ of $\subseteq$-preserving maps from ${}^{<\omega}2$ to ${}^{<\omega_1}2$ such that e° is injective for each $e \in \mathcal{E}$, there is $b \in {}^\omega 2$ such that $F(e^\circ(b)) = 0$ for all $e \in \mathcal{E}$.

We also use conditions [IV] and $[\mathrm{IV}]_{W,X}$. To state these conditions we prepare notation: For a function $F : {}^{<\omega_1}2 \to 2$ and a regular cardinal $\theta \geq \omega_2$ let

$$Y_{F,\theta} := \{M \in [\mathcal{H}_\theta]^\omega \mid \forall B \in ({}^{\omega_1}2) \cap M,\ F(B \restriction (M \cap \omega_1)) = 0\}\ .$$

Let [IV] be the following condition for a function $F : {}^{<\omega_1}2 \to 2$.

[IV] $Y_{F,\theta}$ is stationary for all sufficiently large regular cardinal θ.

Moreover for $W \supseteq \omega_1$ and $X \subseteq [W]^\omega$ let $[\mathrm{IV}]_{W,X}$ be the following condition:

$[\mathrm{IV}]_{W,X}$ For any sufficiently large regular cardinal θ there are stationary many $M \in Y_{F,\theta}$ with $M \cap W \in X$.

Here note that $[\mathrm{IV}]_{W,X}$ implies X to be stationary in $[W]^\omega$ and that if X contains a club set, then $[\mathrm{IV}]_{W,X}$ is equivalent to [IV]. Note also that if F satisfies [II], then all countable $M \prec \langle \mathcal{H}_\theta, \in, F\rangle$ belong to $Y_{F,\theta}$. So [IV] follows from [II], and so does $[\mathrm{IV}]_{W,X}$ if X is stationary.

We claim the following:

Lemma 3.4. *Suppose that $W \supseteq \omega_1$, that $X \subseteq [W]^\omega$ and that $F : {}^{<\omega_1}2 \to 2$ satisfies [III] and $[\mathrm{IV}]_{W,X}$. Then every F-iteration is σ-Baire and forces X to remain stationary in $[W]^\omega$. In particular, if $F : {}^{<\omega_1}2 \to 2$ satisfies [III] and [IV], then any F-iteration is σ-Baire.*

For this we use the following lemma:

Lemma 3.5. *Let $F : {}^{<\omega_1}2 \to 2$ be a function satisfying [III], let $\mathbb{Q}$ be any poset, and let θ be a regular cardinal with $\mathbb{Q} \in \mathcal{H}_\theta$. Suppose that $M \in Y_{F,\theta}$, $M \prec \langle \mathcal{H}_\theta, \in, \mathbb{Q}, F\rangle$ and $q \in \mathbb{Q} \cap M$. Then there is an $(M, \mathbb{Q})$-generic sequence $\vec{q} = \langle q_n \mid n < \omega\rangle$ below q such that $F(u_{\dot{B},\vec{q}}) = 0$ for every $\mathbb{Q}$-name $\dot{B} \in M$ for an $(\omega_1)^V$-branch of $({}^{<\omega_1}2)^V$, where*

$$u_{\dot{B},\vec{q}} := \bigcup\{u \in ({}^{<\omega_1}2) \cap M \mid \exists n < \omega,\ q_n \Vdash \text{“}u \subseteq \dot{B}\text{”}\}\,.$$

Proof. By extending q if necessary, we may assume that q decides whether there is a new $(\omega_1)^V$-branch of $({}^{<\omega_1}2)^V$ in $V^{\mathbb{Q}}$. Let $\alpha := M \cap \omega_1$.

First suppose that q forces that there is no new branch. In this case let $\vec{q} = \langle q_n \mid n < \omega\rangle$ be an arbitrary $(M, \mathbb{Q})$-generic sequence below q. To prove that $\vec{q}$ is as desired, take an arbitrary $\mathbb{Q}$-name $\dot{B} \in M$ for an $(\omega_1)^V$-branch of $({}^{<\omega_1}2)^V$. Then there is $n < \omega$ and $B \in ({}^{\omega_1}2) \cap M$ such that $q_n \Vdash \text{“}\dot{B} = B\text{”}$. Note that $u_{\dot{B},\vec{q}} = B \restriction \alpha$ and that $F(B \restriction \alpha) = 0$ because $B \in M \in Y_{F,\theta}$. So $F(u_{\dot{B},\vec{q}}) = 0$.

Next suppose that q forces that there is a new $(\omega_1)^V$-branch of $({}^{<\omega_1}2)^V$. Let $\mathcal{B}$ be the set of all $\mathbb{Q}$-names $\dot{B} \in M$ which q forces to be such a new branch. Take an enumeration $\langle \dot{B}_n \mid n < \omega\rangle$ of $\mathcal{B}$ such that each element of $\mathcal{B}$ appears in it infinitely many times. Moreover let $\langle \alpha_n \mid n < \omega\rangle$ be an increasing cofinal sequence in α and $\langle A_n \mid n < \omega\rangle$ be an enumeration of all maximal antichains in $\mathbb{Q}$ which belong to M. Then for each $s \in {}^{<\omega}2$ we can inductively take $q_s \in \mathbb{Q} \cap M$ and $u_s \in ({}^{<\omega_1}2) \cap M$ so that (i)–(iv) below hold:

(i) $q_\emptyset \leq q$, and if $s \subseteq t$, then $q_t \leq q_s$.

(ii) q_s meets $A_{\mathrm{dom}(s)}$,

(iii) $\mathrm{dom}(u_s) \geq \alpha_{\mathrm{dom}(s)}$, and $q_s \Vdash$ "$u_s \subseteq \dot{B}_{\mathrm{dom}(s)}$".

(iv) If $s \neq t$, and $\mathrm{dom}(s) = \mathrm{dom}(t)$, then u_s and u_t are incomparable.

For each $b \in {}^{\omega}2$ let $\vec{q}_b := \langle q_{b\restriction n} \mid n < \omega \rangle$. Note that $\vec{q}_b$ is an $(M, \mathbb{Q})$-generic sequence and that

$$u_{\dot{B},\vec{q}_b} \;=\; \bigcup\{u_{b\restriction n} \mid n < \omega \wedge \dot{B} = \dot{B}_n\}$$

for each $\dot{B} \in \mathcal{B}$. For each $\dot{B} \in \mathcal{B}$ define $e_{\dot{B}} : {}^{<\omega}2 \to {}^{<\omega_1}2$ as

$$e_{\dot{B}}(s) \;:=\; \bigcap\{u_t \mid s \subseteq t \in {}^{<\omega}2 \wedge \dot{B}_{\mathrm{dom}(t)} = \dot{B}\}\,.$$

Then each $e_{\dot{B}}$ is $\subseteq$-preserving. Note also that $e_{\dot{B}}(s) = u_s$ if $\dot{B}_{\mathrm{dom}(s)} = \dot{B}$. So $e^{\circ}_{\dot{B}}$ is injective by (iv), and $u_{\dot{B},\vec{q}_b} = e^{\circ}_{\dot{B}}(b)$ for each $b \in {}^{\omega}2$. By [III] we can take $b^* \in {}^{\omega}2$ such that $F(e^{\circ}_{\dot{B}}(b^*)) = 0$ for all $\dot{B} \in \mathcal{B}$. Let $\vec{q} = \langle q_n \mid n < \omega \rangle$ be $\vec{q}_{b^*}$.

We show that $\vec{q}$ is as desired. Take an arbitrary $\mathbb{Q}$-name $\dot{B} \in M$ for an $(\omega_1)^V$-branch of $({}^{<\omega_1}2)^V$. If there is $n < \omega$ with $q_n \Vdash$ "$\dot{B} \in V$", then we can prove that $F(u_{\dot{B},\vec{q}}) = 0$ by the same argument as in the previous case. Otherwise, there is $n < \omega$ with $q_n \Vdash$ "$\dot{B} \notin V$". Then we can take $\dot{B}' \in \mathcal{B}$ such that $q_n \Vdash$ "$\dot{B} = \dot{B}'$". Note that $u_{\dot{B},\vec{q}} = u_{\dot{B}',\vec{q}}$ and that $F(u_{\dot{B}',\vec{q}}) = 0$ by the choice of b^*. So $F(u_{\dot{B},\vec{q}}) = 0$. □

Proof of Lemma 3.4. Let $\vec{\mathbb{Q}} = \langle \mathbb{Q}_\xi, \dot{B}_\eta \mid \xi \leq \zeta, \eta < \zeta \rangle$ be an F-iteration. Suppose that $q \in \mathbb{Q}_\zeta$, that $\mathcal{A}$ is a countable family of maximal antichains in $\mathbb{Q}_\zeta$ and that $\dot{f}$ is a $\mathbb{Q}_\zeta$-name for a function from $[W]^{<\omega}$ to W. It suffices to find $q^* \leq q$ and $x \in X$ such that q^* meets all elements of $\mathcal{A}$ and such that $q^* \Vdash$ "x is closed under $\dot{f}$".

Take a sufficiently large regular cardinal θ. Then there is $M \in Y_{F,\theta}$ such that $x := M \cap W \in X$ and such that $M \prec \langle \mathcal{H}_\theta, \in, F, W, \vec{\mathbb{Q}}, q, \dot{f} \rangle$. We will construct a strongly $(M, \mathbb{Q}_\zeta)$-generic condition q^* below q. Note that q^* will meet all elements of $\mathcal{A}$ because $\mathcal{A} \subseteq M$. Moreover q^* will force x to be closed under $\dot{f}$ by the $(M, \mathbb{Q}_\zeta)$-genericity. So q^* and x will be as desired.

Let $\alpha := M \cap \omega_1$, and let $\vec{q} = \langle q_n \mid n < \omega \rangle$ be an $(M, \mathbb{Q}_\zeta)$-generic sequence below q obtained by applying Lemma 3.5 for $\mathbb{Q}_\zeta$. Note that

$$c_\eta := \bigcup\{q_n(\eta) \mid n < \omega \wedge \eta \in \mathrm{dom}(q_n)\}$$

is a club subset of α for each $\eta \in \zeta \cap M$. Let q^* be the function such that $\mathrm{dom}(q^*) = \bigcup_{n<\omega} \mathrm{dom}(q_n) = \zeta \cap M$ and such that $q^*(\eta) = c_\eta \cup \{\alpha\}$. We claim

that q^* is a strongly $(M, \mathbb{Q}_\zeta)$-generic condition below q. By the construction of q^*, all we have to show is that $q^* \in \mathbb{Q}_\zeta$.

By induction on $\xi \leq \zeta$ we show that $q^* \restriction \xi \in \mathbb{Q}_\xi$. Suppose that $\xi \leq \zeta$ and that $q^* \restriction \xi' \in \mathbb{Q}_{\xi'}$ for all $\xi' < \xi$. If ξ is a limit ordinal, or ξ is a successor ordinal with $\xi - 1 \notin \zeta \cap M$, then $q^* \restriction \xi \in \mathbb{Q}_\xi$ clearly. Suppose that ξ is successor and that $\eta := \xi - 1 \in \zeta \cap M$. Because $q^*(\eta)$ is closed and bounded in ω_1, it suffices to show that $q^* \restriction \eta \Vdash$ "$F(\dot{B}_\eta \restriction \beta) = 0$" for all limit point β of $q^*(\eta)$. This is easily shown for $\beta \in c_\eta$ using the fact that $q^* \restriction \eta \leq q_n \restriction \eta$ and $q_n \in \mathbb{Q}_\zeta$ for all $n < \omega$. Moreover $q^* \restriction \eta \Vdash$ "$u_{\dot{B}_\eta, \vec{q}} = \dot{B}_\eta \restriction \alpha$", and $F(u_{\dot{B}_\eta, \vec{q}}) = 0$ by the choice of $\vec{q}$. So $q^* \restriction \eta \Vdash$ "$F(\dot{B}_\eta \restriction \alpha) = 0$". □

4 Consistency of $\mathsf{PSR}(\mathrm{IA}) + \mathsf{DRP}(\mathrm{IA}) + \Phi$

In this section we prove the following, which implies Theorem 1:

Theorem 4.1. *Assume* MM^{++}. *Then there is a forcing extension in which all* $\mathsf{PSR}(\mathrm{IA})$, $\mathsf{DRP}(\mathrm{IA})$ *and* Φ *hold.*

Note that MM^{++} implies $\mathsf{PSR}(\mathrm{IA})$ and $\mathsf{DRP}(\mathrm{IA})$. We will show that if MM^{++} holds in the ground model, then the following poset $\mathbb{P}$ forces Φ and preserves $\mathsf{PSR}(\mathrm{IA})$ and $\mathsf{DRP}(\mathrm{IA})$:

- $\mathbb{P}$ consists of all partial functions $p : {}^{<\omega_1}2 \to 2$ of size $\leq \omega_1$ such that the set $\{\alpha < \omega_1 \mid B \restriction \alpha \in \mathrm{dom}(p) \wedge p(B \restriction \alpha) = 1\}$ is non-stationary for all $B \in {}^{\omega_1}2$.
- $p_0 \leq p_1$ in $\mathbb{P}$ if $p_0 \supseteq p_1$.

Here note that $\mathbb{P}$ is trivial unless $2^\omega \geq \omega_2$. Let $\dot{F}_\mathbb{P}$ be a $\mathbb{P}$-name for $\bigcup \dot{G}$, where $\dot{G}$ is the canonical name for a $\mathbb{P}$-generic filter. We will prove the following.

Proposition 4.2. *Assume* PFA. *Then* $\dot{F}_\mathbb{P}$ *witnesses* Φ *in* $V^\mathbb{P}$.

Proposition 4.3. *Assume* MM. *Then* $\mathbb{P}$ *forces* $\mathsf{PSR}(\mathrm{IA})$.

Proposition 4.4. *Assume* PFA^{++}. *Then* $\mathbb{P}$ *forces* $\mathsf{DRP}(\mathrm{IA})$.

To prove these propositions we will apply forcing axioms for iterations of $\mathbb{P}$ and several other forcing notions including $\mathbb{Q}(F)$. First we present these auxiliary forcing notions in Subsection 4.1. Then we will prove the above propositions in Subsection 4.2.

4.1 Auxiliary forcing notions

Here we present auxiliary forcing notions, $\mathbb{R}$ and $\mathbb{C}(Z)$, which will be used to prove Proposition 4.2, 4.3 and 4.4.

First we give a forcing notion $\mathbb{R}$, which makes sure that all ω_1-branches of $({}^{<\omega_1}2)^V$ belong to V in some strong sense. $\mathbb{R}$ is based on the forcing notion introduced by Baumgartner [1] and is also used in König-Yoshinobu [8] and Krueger [9].

$\mathbb{R}$ is a three step iteration $\mathrm{Add}(\omega) * \mathrm{Col}(\omega_1, 2^{\omega_1}) * \dot{\mathbb{S}}$, where $\dot{\mathbb{S}}$ is the forcing notion defined as follows: Suppose that G is an $\mathrm{Add}(\omega) * \mathrm{Col}(\omega_1, 2^{\omega_1})$-generic filter over V. Note that $|({}^{\omega_1}2)^V| = \omega_1$ in $V[G]$, so we can easily take a function $\rho : ({}^{\omega_1}2)^V \to \omega_1$ in $V[G]$ such that either $B \restriction \rho(B) \not\subseteq B'$ or $B' \restriction \rho(B') \not\subseteq B$ for any distinct $B, B' \in ({}^{\omega_1}2)^V$. In $V[G]$ let

$$U \;:=\; ({}^{<\omega_1}2)^V \setminus \{B \restriction \alpha \mid B \in ({}^{\omega_1}2)^V \wedge \rho(B) < \alpha < \omega_1\}\,,$$

and let $\mathbb{S}$ be the forcing notion specializing U. That is, $\mathbb{S}$ consists of all finite partial function $p : U \to \omega$ such that $p(u) \neq p(v)$ for any distinct comparable $u, v \in \mathrm{dom}(p)$, and $p \leq q$ in $\mathbb{S}$ if $p \supseteq q$. Let $\dot{\mathbb{S}}$ be an $\mathrm{Add}(\omega) * \mathrm{Col}(\omega_1, 2^{\omega_1})$-name for $\mathbb{S}$.

Here note that $\mathrm{Add}(\omega) * \mathrm{Col}(\omega_1, 2^{\omega_1})$ does not add any new ω_1-branch of $({}^{<\omega_1}2)^V$. This was first noticed by Mitchell [10]. Thus there is no ω_1-branch of U in $V[G]$, and so $\mathbb{S}$ has the c.c.c. in $V[G]$. In particular, $\mathrm{Add}(\omega) * \mathrm{Col}(\omega_1, 2^{\omega_1}) * \dot{\mathbb{S}}$ is proper.

The following summarizes the above mentioned facts on $\mathbb{R}$:

Lemma 4.5. *(1) $\mathbb{R}$ is proper.*

(2) Let $T := {}^{<\omega_1}2$ and $\mathcal{B} := {}^{\omega_1}2$ in V. If G is an $\mathbb{R}$-generic filter over V, then in $V[G]$ there exists a function $\rho : \mathcal{B} \to \omega_1$ and a partial function $\tau : T \to \omega$ such that

(i) either $B \restriction \rho(B) \not\subseteq B'$ or $B' \restriction \rho(B') \not\subseteq B$ for any distinct $B, B' \in \mathcal{B}$,

(ii) $\mathrm{dom}(\tau) = T \setminus \{B \restriction \alpha \mid B \in \mathcal{B} \wedge \rho(B) < \alpha < \omega_1\}$, and $\tau(u) \neq \tau(v)$ for any distinct comparable $u, v \in \mathrm{dom}(\tau)$.

ρ and τ in the above lemma witnesses that all cofinal branch of $({}^{<\omega_1}2)^V$ are in V in the following sense:

Lemma 4.6 (Baumgartner [1])**.** *Suppose that $T \subseteq {}^{<\omega_1}2$ is closed under initial segments and that $\mathcal{B}$ is a set consisting of ω_1-branches of T. Assume that there are a function $\rho : \mathcal{B} \to \omega_1$ and a partial function $\tau : T \to \omega$ satisfying (i) and (ii) in Lemma 4.5. Then all ω_1-branches of T are in $\mathcal{B}$.*

Proof. For the contradiction assume that B is an ω_1-branch of T which is not in $\mathcal{B}$. Then the set $\{\alpha < \omega_1 \mid B \restriction \alpha \in \mathrm{dom}(\tau)\}$ is countable by (ii). Let $\beta < \omega_1$ be an upper bound of this set. Then we can take $B' \in \mathcal{B}$ such that $\rho(B') < \beta$ and $B \restriction \beta = B' \restriction \beta$. Let γ be the least countable ordinal with $B \restriction \gamma \neq B' \restriction \gamma$. Then we can again take $B'' \in \mathcal{B}$ with $\rho(B'') < \gamma$ and $B \restriction \gamma = B'' \restriction \gamma$. Then B' and B'' are distinct elements of $\mathcal{B}$, but $B' \restriction \rho(B') \subseteq B''$ and $B'' \restriction \rho(B'') \subseteq B'$. This contradicts that ρ satisfies (i). □

Next we present $\mathbb{C}(Z)$, which shoots a $\subseteq$-increasing continuous nice $\in$-chain of length ω_1 through a given $Z \subseteq [\mathcal{H}_\lambda]^\omega$. Suppose that λ is a regular cardinal $\geq \omega_2$ and that $Z \subseteq [\mathcal{H}_\lambda]^\omega$. Then let $\mathbb{C}(Z)$ be the following poset:

- $\mathbb{C}(Z)$ consists of all $\subseteq$-increasing continuous nice $\in$-chains $\langle M_\beta \mid \beta \leq \alpha\rangle$ in X for some $\alpha < \omega_1$.
- $p_0 \leq p_1$ in $\mathbb{C}(Z)$ if p_0 is an end-extension of p_1.

We observe basic properties of $\mathbb{C}(Z)$. For a poset $\mathbb{C}$ and a countable set M we say that $\mathbb{C}$ is *M-closed* if every $(M, \mathbb{C})$-generic sequence of length ω has a lower bound. Note that if for some sufficiently large regular cardinal θ there are stationary many $M \in [\mathcal{H}_\theta]^\omega$ with $\mathbb{C}$ M-closed, then $\mathbb{C}$ is σ-Baire. This can be shown by the same argument as in the proof of Lemma 3.4. The following is essentially proved in [4]:

Lemma 4.7 (Feng-Jech [4])**.** *Let λ be a regular cardinal $\geq \omega_2$ and Z be a stationary subset of $[\mathcal{H}_\lambda]^\omega$.*

(1) Suppose that θ is a sufficiently large regular cardinal and that M is a countable elementary submodel of $\langle \mathcal{H}_\theta, \in, \lambda, Z\rangle$ with $M \cap \mathcal{H}_\lambda \in Z$. Then $\mathbb{C}(Z)$ is M-closed.

(2) $\mathbb{C}(Z)$ is σ-Baire.

(3) In $V^{\mathbb{C}(Z)}$ there is a $\subseteq$-increasing continuous cofinal nice $\in$-chain of length ω_1 through Z.

Proof. (2) follows from (1). Note that for any $x \in [\mathcal{H}_\lambda]^\omega$ the set of all $\langle M_\beta \mid \beta \leq \alpha\rangle \in \mathbb{C}(Z)$ with $x \subseteq M_\alpha$ is dense in $\mathbb{C}(Z)$. Then (3) follows from (2) and the definition of $\mathbb{C}(Z)$. So it suffices to prove (1).

Suppose that $\vec{c} = \langle c_n \mid n < \omega\rangle$ is $(M, \mathbb{C}(Z))$-generic sequence. Then it is easy to see that $\bigcup_{n<\omega} c_n$ is of the form $\langle M_\beta \mid \beta < \alpha\rangle$, where α is some countable limit ordinal. (In fact, $\alpha = M \cap \omega_1$.) It is also easily seen that $\bigcup_{\beta<\alpha} M_\beta = M \cap \mathcal{H}_\lambda$. Let $M_\alpha := M \cap \mathcal{H}_\lambda$. Then $\langle M_\beta \mid \beta \leq \alpha\rangle$ is a lower bound of $\{c_n \mid n < \omega\}$. □

4.2 Proof of Theorem 4.1

We will prove Proposition 4.2, 4.3 and 4.4. Here [I]–[IV] denote the conditions of a function $F : {}^{<\omega_1}2 \to 2$ stated in Section 3.

First we give basic properties of $\mathbb{P}$ which can be easily proved:

Lemma 4.8. *Assume that $2^\omega \geq \omega_2$.*

(1) $\mathbb{P}$ is σ-closed.

(2) For any $p \in \mathbb{P}$ and any $U \subseteq {}^{<\omega_1}2$ with $|U| \leq \omega_1$ there is $p' \leq p$ with $\mathrm{dom}(p') \supseteq U$.

Next we observe properties of $\dot{F}_{\mathbb{P}}$:

Lemma 4.9. *Assume that $2^\omega \geq \omega_2$. Then $\dot{F}_{\mathbb{P}}$ is a total function from ${}^{<\omega_1}2$ to 2. Moreover we have the following:*

(1) $\dot{F}_{\mathbb{P}}$ satisfies [I] and [III] in $V^{\mathbb{P}}$.

(2) In V suppose that $W \supseteq \omega_1$ and that $X \subseteq [W]^\omega$ is stationary. Then $\dot{F}_{\mathbb{P}}$ satisfies $[IV]_{W,X}$ in $V^{\mathbb{P}}$. In particular, $\dot{F}_{\mathbb{P}}$ satisfies [IV].

(3) If $\mathbb{P}$ is $<\omega_2$-Baire, then $\dot{F}_{\mathbb{P}}$ satisfies [II] in $V^{\mathbb{P}}$.

Proof. It easily follows from Lemma 4.8 that $\dot{F}_{\mathbb{P}}$ is a total function from ${}^{<\omega_1}2$ to 2 in $V^{\mathbb{P}}$. Moreover (3) is clear from the definition of $\mathbb{P}$ and Lemma 4.8 (2). We prove (1) and (2).

(1) The proof of [I] is similar as and easier than that of [III]. So we only prove [III]. We work in V.

Take an arbitrary $p \in \mathbb{P}$ and an arbitrary sequence $\langle \dot{e}_n \mid n < \omega \rangle$ of $\mathbb{P}$-names such that in $V^{\mathbb{P}}$ each $\dot{e}_n$ is a $\subseteq$-preserving map from ${}^{<\omega}2$ to ${}^{<\omega_1}2$ with $\dot{e}_n^\circ$ injective. It suffices to find $p^* \leq p$ and $b^* \in {}^\omega 2$ such that $p^* \Vdash$ "$\dot{F}_{\mathbb{P}}(\dot{e}_n^\circ(b^*)) = 0$" for all $n < \omega$.

Becasue $\mathbb{P}$ is σ-closed, we can take $p' \leq p$ and a sequence $\langle e_n \mid n < \omega \rangle$ of $\subseteq$-increasing maps from ${}^{<\omega}2$ to ${}^{<\omega_1}2$ such that $p' \Vdash$ "$\dot{e}_n = e_n$" for all $n < \omega$. Note that for each $n < \omega$ there are at most ω_1-many $b \in {}^\omega 2$ with $e_n^\circ(b) \in \mathrm{dom}(p')$ because $|p'| \leq \omega_1$, and e_n° is injective. Then, because $2^\omega \geq \omega_2$, we can take $b^* \in {}^\omega 2$ such that $e_n^\circ(b^*) \notin \mathrm{dom}(p')$ for all $n < \omega$. Let p^* be the extension of p' such that $\mathrm{dom}(p^*) = \mathrm{dom}(p') \cup \{e_n^\circ(b^*) \mid n < \omega\}$ and such that $p^*(e_n^\circ(b^*)) = 0$ for all $n < \omega$. Then p^* and b^* are clearly as desired.

(2) Let θ be a sufficiently large regular cardinal. Take an arbitrary $p \in \mathbb{P}$ and an arbitrary $\mathbb{P}$-name $\dot{f}$ for a function from $[\mathcal{H}_\theta]^{<\omega}$ to $\mathcal{H}_\theta$. It suffices to find $p^* \leq p$ and a $\mathbb{P}$-name $\dot{M}$ such that

$$q^* \Vdash \text{“}\, \dot{M} \in Y_{\dot{F}_{\mathbb{P}},\theta} \wedge \dot{M} \cap W \in X \wedge \dot{M} \text{ is closed under } \dot{f} \,\text{”}\ .$$

Let $\bar{\theta}$ be a sufficiently large regular cardinal, and take a countable $\bar{M} \prec \langle \mathcal{H}_{\bar{\theta}}, \in, X, W, \mathbb{P}, p, \theta, \dot{f} \rangle$ with $\bar{M} \cap W \in X$. Let $\dot{M}$ be a $\mathbb{P}$-name for $\bar{M}[\dot{G}] \cap \mathcal{H}_\theta$. Then $\dot{M}$ is closed under $\dot{f}$ in $V^{\mathbb{P}}$ by the elementarity of $\bar{M}$. Note that any strongly $(\bar{M}, \mathbb{P})$-generic condition forces that $\dot{M} \cap W = \bar{M} \cap W \in X$. Thus it suffices to find a strongly $(\bar{M}, \mathbb{P})$-generic $p^* \leq p$ forcing that $\dot{M} \in Y_{\dot{F}_{\mathbb{P}},\theta}$. The construction of p^* uses almost the same argument as in the proof of Lemma 3.5. By extending p if neccesary we may assume that p decides whether ${}^{\omega_1}2 \subseteq V$. Let $\alpha := \bar{M} \cap \omega_1$.

First assume that $p \Vdash \text{“}\, {}^{\omega_1}2 \subseteq V \,\text{”}$. Then take an $(\bar{M}, \mathbb{P})$-generic sequence $\langle p_n \mid n < \omega \rangle$ below p, and let p^* be a lower bound of $\{p_n \mid n < \omega\}$ by the σ-closure of $\mathbb{P}$. Clearly p^* is a strongly $(\bar{M}, \mathbb{P})$-generic condition below p. To see that $p^* \Vdash \text{“}\, \dot{M} \in Y_{\dot{F}_{\mathbb{P}},\theta} \,\text{”}$, take an arbitrary $\mathbb{P}$-name $\dot{B} \in \bar{M}$ for an element of ${}^{\omega_1}2$. It suffices to show that $p^* \Vdash \text{“}\, \dot{F}_{\mathbb{P}}(\dot{B} \restriction \alpha) = 0 \,\text{”}$ because p^* forces that $\dot{M} \cap \omega_1 = \bar{M} \cap \omega_1 = \alpha$. By the assumption on p we can take $B \in ({}^{\omega_1}2) \cap \bar{M}$ such that $p_n \Vdash \text{“}\, \dot{B} = B \,\text{”}$ for some $n < \omega$. By Lemma 4.8 (2), increasing n if necessary, we may assume that $\{B \restriction \beta \mid \beta < \omega_1\} \subseteq \mathrm{dom}(p_n)$. Then there are club many $\beta < \omega_1$ with $p_n(B \restriction \beta) = 0$ because $p_n \in \mathbb{P}$, and the set of all such β is in $\bar{M}$. Hence $p_n(B \restriction \alpha) = 0$. Thus p^* forces that $\dot{F}_{\mathbb{P}}(B \restriction \alpha) = p_n(B \restriction \alpha) = 0$.

Next assume that $p \Vdash \text{“}\, {}^{\omega_1}2 \not\subseteq V \,\text{”}$. Let $\mathcal{B}$ be the set of all $\mathbb{P}$-names $\dot{B} \in \bar{M}$ such that $p \Vdash \text{“}\, \dot{B} \in {}^{\omega_1}2 \setminus V \,\text{”}$. By exactly the same way as in the proof of Lemma 3.5, we can take $p_s \in \mathbb{P} \cap \bar{M}$ for each $s \in {}^{<\omega}2$ and $e_{\dot{B}} : {}^{<\omega}2 \to {}^{<\omega_1}2$ for each $\dot{B} \in \mathcal{B}$ with the following properties:

(i) For any $b \in {}^{\omega}2$, $\vec{p}_b = \langle p_{b\restriction n} \mid n < \omega \rangle$ is an $(\bar{M}, \mathbb{P})$-generic sequence below p.

(ii) For any $\dot{B} \in \mathcal{B}$, $e_{\dot{B}}$ is $\subseteq$-preserving, and $e^\circ_{\dot{B}}$ is injective.

(iii) For any $\dot{B} \in \mathcal{B}$ and any $b \in {}^{\omega}2$,

$$e^\circ_{\dot{B}}(b) \ = \ \bigcup \{ u \in ({}^{<\omega_1}2) \cap \bar{M} \mid \exists n < \omega,\ p_{b\restriction n} \Vdash \text{“}\, u \subseteq \dot{B} \,\text{”} \}\ .$$

Let $U := \bigcup([{}^{<\omega_1}2]^{\omega_1} \cap \bar{M})$. Then $|U| = \omega_1$. So for each $\dot{B} \in \mathcal{B}$ there are at most ω_1-many $b \in {}^{\omega}2$ with $e^\circ_{\dot{B}}(b) \in U$. Recall also that $2^\omega > \omega_1$ and that $\mathcal{B}$ is countable. Then we can take $b^* \in {}^{\omega}2$ such that $e^\circ_{\dot{B}}(b^*) \notin U$ for all $\dot{B} \in \mathcal{B}$. Let $p' := \bigcup_{n<\omega} p_{b^*\restriction n}$. Note that $\mathrm{dom}(p') = U$. So let p^* be the extension of p' such

that $\mathrm{dom}(p^*) = U \cup \{e^\circ_{\dot B}(b^*) \mid \dot B \in \mathcal{B}\}$ and such that $p^*(e^\circ_{\dot B}(b^*)) = 0$ for each $\dot B \in \mathcal{B}$. Clearly p^* is a strongly $(\bar M, \mathbb{P})$-generic condition below p.

To see that $p^* \Vdash$ "$\dot M \in Y_{\dot F_{\mathbb{P}},\theta}$", take an arbitrary $\mathbb{P}$-name $\dot B \in \bar M$ of elements of ${}^{<\omega_1}2$. We must show that $p^* \Vdash$ "$\dot F_{\mathbb{P}}(\dot B \restriction \alpha) = 0$". If $p_{b^*\restriction n} \Vdash$ "$\dot B \in V$" for some $n < \omega$, then this can be shown by the same argument as in the previous case. Otherwise, there is $n < \omega$ with $p_{b^*\restriction n} \Vdash$ "$\dot B \notin V$". Then we can take $\dot B' \in \mathcal{B}$ with $p^* \Vdash$ "$\dot B = \dot B'$". Note that $p^* \Vdash$ "$\dot B \restriction \alpha = e^\circ_{\dot B'}(b^*)$". So p^* forces that $\dot F_{\mathbb{P}}(\dot B \restriction \alpha) = p^*(e^\circ_{\dot B'}(b^*)) = 0$. □

We proceed to Proposition 4.2:

Proposition 4.2. *Assume* PFA. *Then $\dot F_{\mathbb{P}}$ witnesses Φ in $V^{\mathbb{P}}$.*

By Lemma 4.9 it suffices to prove the following:

Lemma 4.10. *Assume* PFA. *Then $\mathbb{P}$ is $<\omega_2$-Baire.*

Proof. Suppose that $p \in \mathbb{P}$ and that $\mathcal{A}$ is a family of ω_1-many maximal antichains of $\mathbb{P}$. We must find $p^* \le p$ meeting all elements of $\mathcal{A}$. For this we will apply PFA for $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$, where $\dot{\mathbb{Q}}$ and $\dot{\mathbb{R}}$ denote $\mathbb{Q}(\dot F_{\mathbb{P}})$ and a $\mathbb{P} * \dot{\mathbb{Q}}$-name for $\mathbb{R}$, respectively. Note that $\mathbb{P} * \mathbb{Q}$ is proper and σ-Baire by Lemmata 3.4 and 4.9. So $\mathbb{P} * \mathbb{Q} * \dot{\mathbb{R}}$ is also proper by Lemma 4.5.

Take a sufficiently large regular cardinal θ. By Fact 2.1 we can take N and g such that

(i) $N \in [\mathcal{H}_\theta]^{\omega_1}$, and $\omega_1 \subseteq N \prec \langle \mathcal{H}_\theta, \in, \mathbb{P}, \dot{\mathbb{Q}}, \dot{\mathbb{R}}, p, \mathcal{A}\rangle$,

(ii) g is an $(N, \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}})$-generic filter containing $(p, 1_{\dot{\mathbb{Q}}}, 1_{\dot{\mathbb{R}}})$.

Let g_0 and g_1 be the restrictions of g to $\mathbb{P}$ and $\mathbb{P} * \dot{\mathbb{Q}}$, respectively, that is,

$$\begin{aligned} g_0 &:= \{p' \in \mathbb{P} \cap N \mid \exists \dot q, \dot r,\ (p', \dot q, \dot r) \in g\}\ , \\ g_1 &:= \{(p', \dot q) \in (\mathbb{P} * \dot{\mathbb{Q}}) \cap N \mid \exists \dot r,\ (p', \dot q, \dot r) \in g\}\ . \end{aligned}$$

Moreover let $p^* := \bigcup g_0$. Note that if $p^* \in \mathbb{P}$, then p^* extends p and meets all maximal antichains in $\mathcal{A}$ by the genericity of g_0. Below we show that $p^* \in \mathbb{P}$.

Let $T := ({}^{<\omega_1}2) \cap N$, and let $\mathcal{B}$ be the collection of $\dot B_{g_1}$ for all $\mathbb{P} * \dot{\mathbb{Q}}$-name $\dot B \in N$ of an element of ${}^{\omega_1}2$. Then $\mathrm{dom}(p^*) = T$, and for any $B \in \mathcal{B}$ the set $\{\alpha < \omega_1 \mid p^*(B \restriction \alpha) = 1\}$ is non-stationary by an effect of $\dot{\mathbb{Q}}$.

Next let $\dot\rho$ and $\dot\tau$ be $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$-names for ρ and τ in Lem.4.5, and let ρ^* and τ^* be the evaluations of $\dot\rho$ and $\dot\tau$ by g. More precisely, ρ^* is a function from $\mathcal{B}$ to ω_1 such that $\rho^*(\dot B_{g_1}) = \alpha$ if and only if some element of g forces that $\dot\rho(\dot B) = \alpha$. Also, τ^* is a partial function from T to ω such that $\tau^*(u) = n$ if and only if

some element of g forces that $\dot{\tau}(u) = n$. Then using σ-Baireness of $\mathbb{P} * \dot{\mathbb{Q}}$ it is easy to check that ρ^* and τ^* satisfies (i) and (ii) in Lemma 4.5 (2). So all ω_1-branches of T belong to $\mathcal{B}$ by Lemma 4.6. Thus for any ω_1-branch B of T the set $\{\alpha < \omega_1 \mid p^*(B \restriction \alpha) = 1\}$ is non-stationary. Therefore $p^* \in \mathbb{P}$. □

Next we prove Proposition 4.3:

Proposition 4.3. *Assume* MM. *Then* $\mathbb{P}$ *forces* PSR(IA).

We will apply MM to $\mathbb{P} * \mathbb{C}(\dot{Z}) * \mathbb{Q}(\dot{F}_{\mathbb{P}}) * \dot{\mathbb{R}}$, where $\dot{Z}$ is a $\mathbb{P}$-name of a projectively stationary subset of $[\mathcal{H}_\lambda]^\omega$ for some regular $\lambda \geq \omega_2$, and $\dot{\mathbb{R}}$ is a $\mathbb{P} * \mathbb{C}(\dot{Z}) * \mathbb{Q}(\dot{F}_{\mathbb{P}})$-name for $\mathbb{R}$. We need a lemma to see that this four step iteration preserves stationary subsets of ω_1. Here we give it in somewhat a general form for the later use:

Lemma 4.11. *Suppose that* $F : {}^{<\omega_1}2 \to 2$ *satisfies [II] and [III]. Let* $W \supseteq \omega_1$ *and* $X \subseteq [W]^\omega$. *Moreover let* $\mathbb{C}$ *be a forcing notion such that for any sufficiently large regular cardinal* θ *the set*

$$\{M \in [\mathcal{H}_\theta]^\omega \mid M \cap W \in X \wedge \mathbb{C} \text{ is } M\text{-closed}\}$$

is stationary in $[\mathcal{H}_\theta]^\omega$. *Suppose also that* $\dot{\mathbb{Q}}$ *is a* $\mathbb{C}$*-name for an* F*-iteration. Then* $\mathbb{C} * \dot{\mathbb{Q}}$ *is* σ*-Baire and forces* X *to remain stationary in* $[W]^\omega$.

Proof. First note that $\mathbb{C}$ is σ-Baire. Then F satisfies [III] in $V^{\mathbb{C}}$. So by Lemma 3.4 it suffices to show that F satisfies $[\mathrm{IV}]_{W,X}$ in $V^{\mathbb{C}}$. Suppose that θ is a sufficiently large regular cardinal, that $c \in \mathbb{C}$ and that $\dot{f}$ is a $\mathbb{C}$-name for a function from $[\mathcal{H}_\theta]^{<\omega}$ to $\mathcal{H}_\theta$. We find $c^* \leq c$ and a $\mathbb{C}$-name $\dot{M}$ such that

$$c^* \Vdash \text{“}\dot{M} \in Y_{F,\theta} \wedge \dot{M} \cap W \in X \wedge \dot{M} \text{ is closed under } \dot{f}\text{”}\ .$$

Let $\bar{\theta}$ be a sufficiently large regular cardinal. Then we can take a countable $\bar{M} \prec \langle \mathcal{H}_{\bar{\theta}}, \in, F, W, X, \mathbb{C}, c, \theta, \dot{f} \rangle$ such that $\bar{M} \cap W \in X$ and such that $\mathbb{C}$ is $\bar{M}$-closed. Let $\dot{M}$ be a $\mathbb{C}$-name for $\bar{M}[\dot{G}] \cap \mathcal{H}_\theta$, where $\dot{G}$ is the canonical name for a $\mathbb{C}$-generic filter. Next note that $\bar{M} \in Y_{F,\bar{\theta}}$ because F satisfies [II], and $\bar{M} \prec \langle \mathcal{H}_{\bar{\theta}}, \in, F \rangle$. Let $\vec{c} = \langle c_n \mid n < \omega \rangle$ be a $(\bar{M}, \mathbb{C})$-generic sequence below c obtained by applying Lemma 3.5 for $\mathbb{C}$ and $\bar{M}$. Moreover take a lower bound c^* of $\{c_n \mid n < \omega\}$ by the $\bar{M}$-closure of $\mathbb{C}$.

First note that $\dot{M}$ is closed under $\dot{f}$ in $V^{\mathbb{C}}$ by the elementarity of $\bar{M}$. Moreover c^* forces that $\dot{M} \cap W = \bar{M} \cap W \in X$ by the $(\bar{M}, \mathbb{C})$-genericity. Finally c^* forces that $\dot{M} \cap \omega_1 = \bar{M} \cap \omega_1 =: \alpha$. Moreover, by its choice, c^* forces that $F(\dot{B} \restriction \alpha) = 0$ for every $\mathbb{C}$-name $\dot{B} \in M$ for an element of ${}^{\omega_1}2$. Thus c^* forces that $\dot{M} \in Y_{F,\theta}$. Therefore c^* and $\dot{M}$ are as desired. □

From the lemma above we obtain the following lemma. (1) follows immediately from the above lemma and Lemma 4.7. For (2) note that if $\mathbb{C}$ is σ-closed, then $\mathbb{C}$ is M-closed for all countable set M:

Lemma 4.12. *Suppose that $F : {}^{<\omega_1}2 \to 2$ satisfies [II] and [III].*

(1) Let Z be a projectively stationary subset of $[\mathcal{H}_\lambda]^\omega$ for some regular $\lambda \geq \omega_2$, and let $\dot{\mathbb{Q}}$ be a $\mathbb{C}(Z)$-name for an F-iteration. Then $\mathbb{C}(Z) * \dot{\mathbb{Q}}$ preserves stationary subsets of ω_1 and is σ-Baire.

(2) Let $\mathbb{C}$ be a σ-closed poset, and let $\dot{\mathbb{Q}}$ be a $\mathbb{C}$-name for an F-iteration. Then $\mathbb{C} * \dot{\mathbb{Q}}$ is proper and σ-Baire.

Now we prove Proposition 4.3:

Proof of Proposition 4.3. Suppose that $p \in \mathbb{P}$. Let λ be a regular cardinal $\geq \omega_2$, and let $\dot{Z}$ be a $\mathbb{P}$-name for a projectively stationary subset of $[\mathcal{H}_\lambda]^\omega$. We will find $p^* \leq p$ which forces that $\dot{Z}$ contains an $\subseteq$-increasing continuous nice $\in$-chain of length ω_1.

Let $\dot{\mathbb{C}}$ and $\dot{\mathbb{Q}}$ denote $\mathbb{C}(\dot{Z})$ and $\mathbb{Q}(\dot{F}_{\mathbb{P}})$, respectively, and let $\dot{\mathbb{R}}$ be a $\mathbb{P} * \dot{\mathbb{C}} * \dot{\mathbb{Q}}$-name for $\mathbb{R}$. Here note that $\dot{F}_{\mathbb{P}}$ satisfies [II] and [III] in $V^{\mathbb{P}}$ by Lemmata 4.9 and 4.10. Hence $\dot{\mathbb{C}} * \dot{\mathbb{Q}}$ preserves stationary subsets of ω_1 and is σ-Baire by Lemma 4.12 (1). So $\mathbb{P} * \dot{\mathbb{C}} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ preserves stationary subsets of ω_1 by the properness of $\mathbb{P}$ and $\mathbb{R}$. Let θ be a sufficiently large regular cardinal. By Fact 2.1 we can take N and g such that

(i) $N \in [\mathcal{H}_\theta]^{\omega_1}$, and $\omega_1 \subseteq N \prec \langle \mathcal{H}_\theta, \in, \mathbb{P}, \dot{Z}, \dot{\mathbb{C}}, \dot{\mathbb{Q}}, \dot{\mathbb{R}}, p \rangle$,

(ii) g is an $(N, \mathbb{P} * \dot{\mathbb{C}} * \dot{\mathbb{Q}} * \dot{\mathbb{R}})$-generic filter containing $(p, 1_{\dot{\mathbb{C}}}, 1_{\dot{\mathbb{Q}}}, 1_{\dot{\mathbb{R}}})$.

Let g_0 be the restrictions of g to $\mathbb{P}$, and let $p^* := \bigcup g_0$. Then $p^* \in \mathbb{P}$ by the same argument as in the proof of Lemma 4.10. Note that $p^* \leq p$.

To show that p^* is as desired, suppose that G_0 is a $\mathbb{P}$-generic filter over V which contains p^*, and let Z be $\dot{Z}_{G_0}$. In $V[G_0]$ we find a $\subseteq$-increasing continuous nice $\in$-chain through Z. First note that $N[G_0] \prec \langle \mathcal{H}_\theta, \in, Z \rangle$. Moreover, using the genericity of g and the fact that $g_0 \subseteq G_0$, it is easy to check that

$$h \ := \ \{\dot{c}_{G_0} \mid \dot{c} \in N \wedge \exists p, \dot{q}, \dot{r},\ (p, \dot{c}, \dot{q}, \dot{r}) \in g\}$$

is an $(N[G_0], \mathbb{C}(Z))$-generic filter. Then $\bigcup h$ is a $\subseteq$-increasing continuous nice $\in$-chain through Z. $\square$

Finally we will prove Proposition 4.4:

Proposition 4.4. *Assume* PFA^{++}. *Then* $\mathbb{P}$ *forces* $\mathsf{DRP}(\mathrm{IA})$.

Proof. Suppose that $p \in \mathbb{P}$, that λ and $\bar{\lambda}$ are regular cardinals with $\omega_2 \leq \lambda < \bar{\lambda}$, and that $\dot{f}$ is a $\mathbb{P}$-name for a function from $[\mathcal{H}_{\bar{\lambda}}]^{<\omega}$ to $\mathcal{H}_{\bar{\lambda}}$. It suffices to find $p^* \leq p$ which forces that there is $N^* \in [\mathcal{H}_{\bar{\lambda}}]^{\omega_1}$ such that

(i) $\omega_1 \subseteq N^*$, and N^* is closed under $\dot{f}$,

(ii) N^* is internally approachable of length ω_1,

(iii) $X \cap [N^* \cap \mathcal{H}_\lambda]^\omega$ is stationary for any stationary $X \subseteq [\mathcal{H}_\lambda]^\omega$ in N^*.

Let $\dot{\mathbb{C}}$ be a $\mathbb{P}$-name for $\mathrm{Col}(\omega_1, \mathcal{H}_{\bar{\lambda}})$, let $\dot{\mathbb{Q}}$ denote $\mathbb{Q}(\dot{F}_{\mathbb{P}})$ in $V^{\mathbb{P}*\dot{\mathbb{C}}}$, and let $\dot{\mathbb{R}}$ be a $\mathbb{P} * \dot{\mathbb{C}} * \dot{\mathbb{Q}}$-name for $\mathbb{R}$. As in the proof of the previous proposition, $\dot{F}_{\mathbb{P}}$ satisfies [II] and [III] by Lemmata 4.9 and 4.10. Thus $\dot{\mathbb{C}} * \dot{\mathbb{Q}}$ is proper in $V^{\mathbb{P}}$ by Lemma 4.12 (2). So $\mathbb{P} * \dot{\mathbb{C}} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is proper. Take a sufficiently large regular cardinal θ. By Fact 2.1 we can take N and g such that

(iv) $N \in [\mathcal{H}_\theta]^{\omega_1}$, and $\omega_1 \subseteq N \prec \langle \mathcal{H}_\theta, \in, \mathbb{P}, \dot{\mathbb{C}}, \dot{\mathbb{Q}}, \dot{\mathbb{R}}, \lambda, \bar{\lambda}, p, \dot{f} \rangle$,

(v) g is an $(N, \mathbb{P} * \dot{\mathbb{C}} * \dot{\mathbb{Q}} * \dot{\mathbb{R}})$-generic filter containing $(p, 1_{\dot{\mathbb{C}}}, 1_{\dot{\mathbb{Q}}}, 1_{\dot{\mathbb{R}}})$,

(vi) $\dot{S}_g$ is stationary for every $\mathbb{P} * \dot{\mathbb{C}} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$-name $\dot{S} \in N$ for a stationary subset of ω_1.

Let g_0 be the restriction of g to $\mathbb{P}$, and let p^* be $\bigcup g_0$. Then $p^* \in \mathbb{P}$ by the same argument as in the proof of Proposition 4.2, and $p^* \leq p$.

We show that p^* is as desired. Suppose that G_0 is a $\mathbb{P}$-generic filter over V containing p^*, and in $V[G_0]$ let $N^* := N[G_0] \cap \mathcal{H}_{\bar{\lambda}}$. Then N^* clearly satisfies (i). We must check (ii) and (iii). Below let $\dot{H}$ be the canonical $\mathbb{P} * \dot{\mathbb{C}}$-name for a $\dot{\mathbb{C}}$-generic filter over $V^{\mathbb{P}}$.

First we check (ii). Let π be the interpretation of $\bigcup \dot{H}$ by g, that is,

$$\pi \ := \ \bigcup \{ \dot{c}_{G_0} \mid \exists p, \dot{q}, \dot{r}, \ (p, \dot{c}, \dot{q}, \dot{r}) \in g \} \ .$$

Note that π is a surjection from ω_1 to N^* and that all proper initial segments belong to N^*. Then we can inductively construct an increasing cofinal sequence $\langle \gamma_\alpha \mid \alpha < \omega_1 \rangle$ in ω_1 so that $\langle \pi[\gamma_\alpha] \mid \alpha < \omega_1 \rangle$ is a nice $\in$-chain. Because $\bigcup_{\alpha < \omega_1} \pi[\gamma_\alpha] = \pi[\omega_1] = N^*$, it follows that N^* is internally approachable of length ω_1.

Next we check (iii). Take an arbitrary stationary $X \subseteq [\mathcal{H}_\lambda]^\omega$ in N^*. Let $\dot{X} \in N$ be a $\mathbb{P}$-name for X, and let $\dot{S} \in N$ be a $\mathbb{P} * \dot{\mathbb{C}}$-name for the set

$$\{ \alpha < \omega_1 \mid (\textstyle\bigcup \dot{H})[\alpha] \cap (\mathcal{H}_\lambda)^{V^{\mathbb{P}}} \in \dot{X} \} \ .$$

Here recall that $\dot{\mathbb{C}} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ is proper in $V^{\mathbb{P}}$. So $\dot{X}$ remains stationary in $V^{\mathbb{P}*\dot{\mathbb{C}}*\dot{\mathbb{Q}}*\dot{\mathbb{R}}}$. Then by the continuity of the map $\alpha \mapsto (\bigcup \dot{H})[\alpha]$, $\mathbb{P}*\dot{\mathbb{C}}*\dot{\mathbb{Q}}*\dot{\mathbb{R}}$ forces $\dot{S}$ stationary. Thus $\dot{S}_g$ is a stationary subset of ω_1 in V, and so is in $V[G_0]$ because $\mathbb{P}$ is σ-closed. Here note that

$$X \cap [N^* \cap \mathcal{H}_\lambda]^\omega \ \supseteq \ \{\pi[\alpha] \cap \mathcal{H}_\lambda \mid \alpha \in \dot{S}_g\}$$

by the construction of π. Thus $X \cap [N^* \cap \mathcal{H}_\lambda]^\omega$ is stationary. □

5 Consistency of $\mathsf{DRP}(\mathrm{IA}) + \mathsf{CH} + \Phi$

In this section we prove the following, which implies Theorem 2:

Theorem 5.1. *Assume that there is a supercompact cardinal. Then there exists a forcing extension in which all* $\mathsf{DRP}(\mathrm{IA})$, CH *and* Φ *hold.*

As in the previous section, [I]–[IV] denote the properties of a function $F : {}^{<\omega_1}2 \to 2$ stated in Section 3. We use the following:

Lemma 5.2. *Let* κ *be an inaccessible cardinal. Then* $\mathrm{Col}(\omega_1, <\kappa)$ *adds a function* $F : {}^{<\omega_1}2 \to 2$ *satisfying [I], [III] and [IV].*

Proof. Let $\mathbb{P}$ be the poset of all countable partial functions $p : {}^{<\omega_1}2 \to 2$ ordered by reverse inclusions. First note that if CH holds, then $|{}^{<\omega_1}2| = \omega_1$, and so $\mathbb{P}$ is forcing equivalent to $\mathrm{Col}(\omega_1, 2)$. Moreover $\mathrm{Col}(\omega_1, <\kappa)$ forces CH. So

$$\mathrm{Col}(\omega_1, <\kappa) \ \sim \ \mathrm{Col}(\omega_1, <\kappa) * \mathrm{Col}(\omega_1, 2) \ \sim \ \mathrm{Col}(\omega_1, <\kappa) * \dot{\mathbb{P}}\,,$$

where $\dot{\mathbb{P}}$ denotes a $\mathrm{Col}(\omega_1, <\kappa)$-name for $\mathbb{P}$, and $\sim$ denotes the forcing equivalence. Hence it suffices to show that $\mathbb{P}$ adds F as in the lemma.

Let $\dot{G}$ be the canonical name for a $\mathbb{P}$-generic filter, and let $\dot{F}$ be a $\mathbb{P}$-name for $\bigcup \dot{G}$. By the same argument as in the proof of Lemma 4.9 we can prove that $\dot{F}$ satisfies [I] and [III] in $V^{\mathbb{P}}$. We show that $\dot{F}$ satisfies [IV] in $V^{\mathbb{P}}$. Let θ be a regular cardinal $\geq \omega_2$ which remains regular in $V^{\mathbb{P}}$. Suppose that $p \in \mathbb{P}$ and that $\dot{f}$ is a $\mathbb{P}$-name for a function from $[\mathcal{H}_\theta]^{<\omega}$ to $\mathcal{H}_\theta$. It suffices to find $p^* \leq p$ and a $\mathbb{P}$-name $\dot{M}$ such that

$$p^* \Vdash \text{“}\, \dot{M} \in Y_{\dot{F},\theta} \wedge \dot{M} \text{ is closed under } \dot{f} \,\text{”}\,.$$

Take a sufficiently large regular $\bar{\theta}$ and a countable $\bar{M} \prec \langle \mathcal{H}_{\bar{\theta}}, \in, \mathbb{P}, p, \theta, \dot{f} \rangle$. Let $\dot{M}$ be a $\mathbb{P}$-name for $\bar{M}[\dot{G}] \cap \mathcal{H}_\theta$. Note that $\dot{M}$ is closed under $\dot{f}$ in $V^{\mathbb{P}}$ by the elementarity of $\bar{M}$. Next take a $(\bar{M}, \mathbb{P})$-generic sequence $\langle p_n \mid n < \omega \rangle$

below p, and let $p' := \bigcup_{n<\omega} p_n$. Moreover let $\alpha := \bar{M} \cap \omega_1$, and let $\mathcal{B}$ be the set of all $\mathbb{P}$-name $\dot{B} \in \bar{M}$ for an element of ${}^{\omega_1}2$. Then for each $\dot{B} \in \mathcal{B}$ there is $u_{\dot{B}} \in {}^{\alpha}2$ such that $p' \Vdash$ "$\dot{B} \restriction \alpha = u_{\dot{B}}$". Note that $u_{\dot{B}} \notin \mathrm{dom}(p')$ because $\mathrm{dom}(p') \subseteq ({}^{<\omega_1}2) \cap \bar{M} \subseteq {}^{<\alpha}2$. Let p^* be the extension of p' such that $\mathrm{dom}(p^*) = \mathrm{dom}(p') \cup \{u_{\dot{B}} \mid \dot{B} \in \mathcal{B}\}$ and such that $p^*(u_{\dot{B}}) = 0$ for all $\dot{B} \in \mathcal{B}$. Clearly $p^* \in \mathbb{P}$, and $p^* \leq p$.

We must show that $p^* \Vdash$ "$\dot{M} \in Y_{\dot{F},\theta}$". First $p^* \Vdash$ "$\dot{M} \cap \omega_1 = \bar{M} \cap \omega_1 = \alpha$" by the genericity of p^*. Moreover, by its construction, p^* forces that $\dot{F}(\dot{B} \restriction \alpha) = 0$ for all $\dot{B} \in \mathcal{B}$. Thus p^* forces that $\dot{M} \in Y_{\dot{F},\theta}$. □

Proof of Theorem 5.1. Let κ be a supercompact cardinal. Moreover let $\dot{F}$ be a $\mathrm{Col}(\omega_1, <\kappa)$-name of F as in Lemma 5.2. Then $\mathbb{Q}(\dot{F})$ is σ-Baire in $V^{\mathrm{Col}(\omega_1,<\kappa)}$ by Lemma 3.4. So $\mathrm{Col}(\omega_1, <\kappa) * \mathbb{Q}(\dot{F})$ forces that $\dot{F}$ witnesses Φ by Lemma 3.3 and that CH holds. Thus it suffices to prove that $\mathrm{Col}(\omega_1, <\kappa) * \mathbb{Q}(\dot{F})$ forces DRP(IA). Before starting, we make a remark on the proof below. We will deal with several transitive models of ZFC, and the argument below can be carried out in some large forcing extension of V, for example, an extension of V by $\mathrm{Col}(\omega, \chi)$ for some enough large cardinal χ. Note also that $[\mathrm{On}]^{\omega}$ is absolute among all models appearing below. In particular, ω_1, $\mathrm{Col}(\omega_1, <\kappa)$, etc. are absolute.

Take a $\mathrm{Col}(\omega_1, <\kappa)$-generic filter G_0 over V, and let $V' := V[G_0]$ and $F := \dot{F}_{G_0}$. Note that $\mathbb{Q}(F)$ is an F-iteration of length κ in V'. Let $\vec{\mathbb{Q}} = \langle \mathbb{Q}_\xi, \dot{B}_\eta \mid \xi \leq \kappa, \eta < \kappa \rangle$ be an F-iteraiton with $\mathbb{Q}(F) = \mathbb{Q}_\kappa$. Take a $\mathbb{Q}_\kappa$-generic filter H_0 over V', and let $V'' := V'[H_0]$. In V'', suppose that λ and $\bar{\lambda}$ are regular cardinals with $\kappa \leq \lambda < \bar{\lambda}$, and let W and $\bar{W}$ be $\mathcal{H}_\lambda$ and $\mathcal{H}_{\bar{\lambda}}$, respectively. Morever take an arbitrary function $f : [\bar{W}]^{<\omega} \to \bar{W}$. It suffices to show that in V'' there is $\bar{N} \in [\bar{W}]^{\omega_1}$ with the following properties:

(i) $\omega_1 \subseteq \bar{N}$, and $\bar{N}$ is closed under f.

(ii) $\bar{N}$ is internally approachable of length ω_1.

(iii) $X \cap [\bar{N} \cap W]^{\omega}$ is stationary for all stationary $X \subseteq [W]^{\omega}$ with $X \in \bar{N}$.

For this we will use a generic elementary embedding.

In V let θ be a sufficiently large cardinal, and take an elementary embedding $j : V \to K$ witnessing that κ is θ-supercompact. Note that $j(\mathrm{Col}(\omega_1, <\kappa))$ is equal to $\mathrm{Col}(\omega_1, <\kappa) \times \mathrm{Col}(\omega_1, \kappa, <j(\kappa))$. Let G_1 be a $\mathrm{Col}(\omega_1, \kappa, <j(\kappa))$-generic filter over $K[G_0 * H_0]$, and let $K' := K[G_0 * G_1]$. Then by the standard argument j can be extended to an elementary embedding from V' to K'. For simplicity of our notation, this elementary embedding is also denoted as j. Note that $\vec{\mathbb{Q}}$ is

an initial segment of $j(\vec{\mathbb{Q}})$. Hence $j(\mathbb{Q}_\kappa)$ can be decomposed to $\mathbb{Q}_\kappa * \dot{\mathbb{Q}}_{\rm tail}$ in K'. Note that H_0 is $\mathbb{Q}_\kappa$-generic over K'. Let H_1 be a $(\dot{\mathbb{Q}}_{\rm tail})_{H_0}$-generic filter over $K'[H_0]$. Then, because $\mathbb{Q}_\kappa \subseteq \mathcal{H}_\kappa$ in V', j can be extended to an elementary embedding $j : V'' \to K''$.

Let $\bar{N} := j[\bar{W}]$. Note that $j \restriction \bar{W} \in K''$ because it can be recovered from $j \restriction (\mathcal{H}_{\bar{\lambda}})^V$, G_0, G_1, H_0 and H_1, which are all in K''. Note also that $|\bar{W}| = \omega_1$ in K''. So $\bar{N} \in [j(\bar{W})]^{\omega_1}$ in K''. Then by the elementarity of j it suffices for (i)–(iii) in V'' to prove that the following hold in K'':

(iv) $\omega_1 \subseteq \bar{N}$, and $\bar{N}$ is closed under $j(f)$.

(v) $\bar{N}$ is internally approachable of length ω_1.

(vi) $X \cap [j[W]]^\omega$ is stationary for all stationary $X \subseteq [j(W)]^\omega$ with $X \in \bar{N}$.

Here note that $j[W] = \bar{N} \cap j(W)$ in (vi).

(iv) is clear from the construction of $\bar{N}$ and the elementarity of j. For (v) note that $\bar{W}$ is closed under countable sequences in K''. Then so is $\bar{N}$ by the elementarity of j, and (v) easily follows from this fact. We will check (vi) below:

In K'' take an arbitrary stationary $X \subseteq [j(W)]^\omega$ in $\bar{N}$. Let $Y := j^{-1}(X)$. Then Y is stationary subset of $[W]^\omega$ in $V'' = V[G_0 * H_0]$. Here note that $({}^\theta K[G_0 * H_0]) \cap V'' \subseteq K[G_0 * H_0]$ because $({}^\theta K) \cap V \subseteq K$, and the size of $\mathrm{Col}(\omega_1, < \kappa) * \mathbb{Q}(\dot{F})$ is less than θ in V. So $Y \in K[G_0 * H_0]$. Moreover Y is stationary in $[W]^\omega$ in $K[G_0 * H_0]$ because $K[G_0 * H_0] \subseteq V''$. Here note that K'' is a generic extension of $K[G_0 * H_0]$ by $\mathrm{Col}(\omega_1, \kappa, < j(\kappa)) * (\dot{\mathbb{Q}}_{\rm tail})_{H_0}$. Note also that F satisfies [II] and [III] in $K[G_0 * H_0]$. Hence Y remains stationary in $[W]^\omega$ in K'' by Lemma 4.12 (2). Then $Z := \{j[y] \mid y \in Y\}$ is stationary in $[j[W]]^\omega$ in K'' because $j \restriction W : W \to j[W]$ is a bijection. But $j[y] = j(y)$ for each $y \in Y$ because y is countable. Hence $Z = j[Y] \subseteq X \cap [j[W]]^\omega$. So $X \cap [j[W]]^\omega$ is stationary in K''. □

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