Generalized Prikry forcing and iteration of generic ultrapowers

Hiroshi Sakai

1 Introduction

It is known that there is a close relation between Prikry forcing and the iteration of ultrapowers:

Theorem 1.1. (Solovay) *Assume κ is a measurable cardinal and U is a normal ultrafilter on κ. Let $\langle M_n, j_{m,n} \mid m \leq n \leq \omega \rangle$ be the iteration of ultrapowers of V by U. Then the sequence $\langle j_{0,n}(\kappa) \mid n \in \omega \rangle$ is a Prikry generic sequence over M_ω with respect to $j_{0,\omega}(U)$.*

Moreover Bukovský [1] and Dehornoy [2] showed that the generic extension $M_\omega[\langle j_{0,n}(\kappa) \mid n \in \omega \rangle]$ is $\bigcap_{n \in \omega} M_n$ in Theorem 1.1. (For the history of these results, read the introduction of Dehornoy [2] and pp.259-260 of Kanamori [6].) In Dehornoy [3], these results were generalized for the forcing of Magidor [7] which changes a measurable cardinal of higher Mitchell order into a singular cardinal of uncountable cofinality.

In this paper we generalize Theorem 1.1 for normal filters which are not necessarily maximal. Of course, the above theorem can be restated using the dual ideal of U. In this paper we argue using ideals instead of filters. Assume κ is regular uncountable and I is a normal precipitous ideal on κ.

Prikry forcing has two natural generalizations, PR^*_I and PR^+_I. PR^*_I consists of all pairs $\langle t, T \rangle$ such that $t \in {}^{<\omega}\kappa$ and $T \subseteq {}^{<\omega}\kappa$ is a tree in which every node has I-measure 1 immediate successors, i.e. $\{\alpha < \kappa \mid s^\frown\langle \alpha \rangle \in T\}$ is of I-measure 1 for every $s \in T$. PR^+_I consists of all pairs $\langle t, T \rangle$ such that $t \in {}^{<\omega}\kappa$ and $T \subseteq {}^{<\omega}\kappa$ is a tree in which every node has I-positive immediate successors. In both PR^*_I and PR^+_I, the order is defined as $\langle t_0, T_0 \rangle \leq \langle t_1, T_1 \rangle$ if and only if for every $s_0 \in T_0$, there is an $s_1 \in T_1$ such that $t_0{}^\frown s_0 = t_1{}^\frown s_1$. Note that if I is maximal then PR^*_I and PR^+_I coincide and this is the tree type Prikry forcing notion. Note also that if $\kappa = \omega_2$ and I is the ideal of bounded subsets of ω_2 then PR^+_I is the Namba forcing notion. So PR^+_I is often treated as a variant of Namba forcing.

On the other hand, the iteration of ultrapowers has an obvious generalization, the iteration of generic ultrapowers. Unlike the iteration of ultrapowers, the iteration of generic ultrapowers is not uniquely determined by I. It depends on the choice of generic filters by which the ultrapowers are constructed.

We generalize Theorem 1.1 for both PR_I^* and PR_I^+. In Theorem 3.3, we show that for some kind of sequence of generic filters $\langle G_n \mid n \in \omega \rangle$, letting $\langle M_n, G_m, j_{m,n} \mid m \leq n \leq \omega, m < \omega \rangle$ be the iteration of generic ultrapowers of V by $\langle G_n \mid n \in \omega \rangle$, the sequence of critical points $\langle j_{0,n}(\kappa) \mid n \in \omega \rangle$ is a generic sequence for $PR^*_{j_{0,\omega}(I)}$ over M_ω. In Theorem 3.5, we show that for another kind of sequence of generic filters, the sequence of critical points is a generic sequence for $PR^+_{j_{0,\omega}(I)}$ over M_ω.

In Section 2, we review basics on the iteration of generic ultrapowers. Section 3 is the main part of this paper. We show Theorem 3.3 and 3.5. After that, we observe some known facts on PR^* and PR^+ from the point of view of Theorem 3.3 and 3.5.

Notations and Facts:

In general we follow notations of Kanamori [6].

First we give notations and a fact related to the generic ultrapower and its iteration. For an ideal I on some set X, I^+ is the set of all I-positive sets, I^* is the dual filter of I and $\mathbb{P}_I$ is the poset $\langle I^+, \subseteq \rangle$.

Let M be a transitive model of ZFC and $X, I \in M$ be such that $M \models$ "I is an ideal on X". Let G be an $(M, (\mathbb{P}_I)^M)$-generic filter. Then for each function $f \in M$ on X, $(f)_G$ denotes the element of $Ult(M, G)$ represented by f, where $Ult(M, G)$ denotes the ultrapower of M by G. If $Ult(M, G)$ is well-founded then $[f]_G$ denotes the corresponding element of $(f)_G$ in the transitive collapse of $Ult(M, G)$. We say that $j : M \to N \cong Ult(M, G)$ is the generic ultrapower map associated with G if N is the transitive collapse of $Ult(M, G)$ and $j(a) = [c_a]_G$ for each $a \in M$, where c_a is the constant function on X with its value a.

Let M, X, I be as above and let $\alpha \in On$. A sequence $\langle M_\xi, G_\eta, j_{\eta,\xi} \mid \eta \leq \xi < \alpha, \eta + 1 < \alpha \rangle$ is called an *iteration of generic ultrapowers of M by I of length α* if the following holds:

- $M_0 = M$ and M_ξ is a transitive model of ZFC for each ξ.
- $j_{\eta,\xi} : M_\eta \to M_\xi$ is an elementary embedding for each η, ξ and $\langle M_\xi, j_{\eta,\xi} \mid \eta \leq \xi < \alpha, \eta + 1 < \alpha \rangle$ is a commutative system.
- For each η, G_η is $(M_\eta, j_{0,\eta}((\mathbb{P}_I)^M))$-generic and $j_{\eta,\eta+1} : M_\eta \to M_{\eta+1} \cong Ult(M_\eta, G_\eta)$ is the generic ultrapower map.
- If $\beta < \alpha$ is a limit ordinal then M_β is the transitive collapse of the direct limit of $\langle M_\xi, j_{\eta,\xi} \mid \eta \leq \xi < \beta \rangle$ and $j_{\eta,\beta} : M_\eta \to M_\beta$ is the induced map.

We call M_ξ the *ξ-th iterate of generic ultrapowers of M by I*. In Woodin [12] (Lemma 3.10. and Remark 3.11.), it is remarked that if I is precipitous in M and $\xi \in M \cap On$ then well-foundedness of the ξ-th iterate is always guaranteed:

Fact 1.2. *Let M be a transitive model of ZFC and $X, I \in M$ be such that $M \models$ "I is a precipitous ideal on X". Let $\alpha < On \cap M$ and $\langle M_\xi, H_\eta, j_{\eta,\xi} \mid \eta \leq \xi < \alpha, \eta + 1 < \alpha \rangle$ be an iteration of generic ultrapowers of M by I. Then:*

(1) *If α is a successor ordinal, say $\alpha = \beta+1$, and G is a $(M_\beta, j_{0,\beta}(\mathbb{P}_I))$-generic filter then $Ult(M_\beta, G)$ is well-founded.*

(2) *If α is a limit ordinal then the direct limit of $\langle M_\xi, j_{\eta,\xi} \mid \eta \leq \xi < \alpha \rangle$ is well-founded.*

Next we give notations about embeddings between posets. Let $\mathbb{P}$ and $\mathbb{Q}$ be posets.

$\sigma : \mathbb{P} \to \mathbb{Q}$ is a complete embedding if

(1) σ is order preserving, i.e. $\forall p_1, p_2 \in \mathbb{P}$, $p_1 \leq p_2 \to \sigma(p_1) \leq \sigma(p_2)$,

(2) for each maximal antichain $A \subseteq \mathbb{P}$, $\sigma[A]$ is a maximal antichain of $\mathbb{Q}$.

$\pi : \mathbb{Q} \to \mathbb{P}$ is a projection if

(1) π is surjective and order preserving,

(2) $\forall q \in \mathbb{Q} \forall p \in \mathbb{P}$, if $p \leq \pi(q)$ then there is a $q^* \leq q$ such that $\pi(q^*) = p$.

Projections which appear in this paper have the following additional property:

(3) $\forall q \in \mathbb{Q} \forall p \in \mathbb{P}$, if $p \geq \pi(q)$ then there is a $q^* \geq q$ such that $\pi(q^*) = p$.

We call π a good projection if π satisfies (1)-(3).

If $\sigma : \mathbb{P} \to \mathbb{Q}$ is a complete embedding and G is $\mathbb{P}$-generic then the quotient $\mathbb{Q}/_\sigma G$ is the poset obtained from restricting $\mathbb{Q}$ to $\{q \in \mathbb{Q} \mid \forall p \in G, q \text{ is compatible with } \sigma(p)\}$. If $\pi : \mathbb{Q} \to \mathbb{P}$ is a projection and G is $\mathbb{P}$-generic then the quotient $\mathbb{Q}/_\pi G$ is the poset obtained from restricting $\mathbb{Q}$ to $\pi^{-1}[G]$. If σ or π is clear from the context, we just write $\mathbb{Q}/G$ for $\mathbb{Q}/_\sigma G$ or $\mathbb{Q}/_\pi G$.

The following is basic.

Fact 1.3. *Let $\mathbb{P}$ and $\mathbb{Q}$ be posets.*

(1) *Assume that $\sigma : \mathbb{P} \to \mathbb{Q}$ is a complete embedding. Then H is $(V, \mathbb{Q})$-generic iff $G := \sigma^{-1}[H]$ is $(V, \mathbb{P})$-generic and H is $(V[G], \mathbb{Q}/_\sigma G)$-generic.*

(2) *Assume that $\pi : \mathbb{Q} \to \mathbb{P}$ is a good projection. Then H is $(V, \mathbb{Q})$-generic iff $G := \pi[H]$ is $(V, \mathbb{P})$-generic and H is $(V[G], \mathbb{Q}/_\pi G)$-generic.*

(3) *Assume that $\sigma : \mathbb{P} \to \mathbb{Q}$ is a complete embedding, $\pi : \mathbb{Q} \to \mathbb{P}$ is a projection and $\pi \circ \sigma = id$.*

   (a) *If H is $(V, \mathbb{Q})$-generic then $\sigma^{-1}[H] = \pi[H]$.*

   (b) *If G is $(V, \mathbb{P})$-generic then $\mathbb{Q}/_\sigma G = \mathbb{Q}/_\pi G$.*

2 Iteration of generic ultrapowers.

In this section, we review basics on the iteration of generic ultrapowers. This is a natural generalization of Kunen's theory of iterated ultrapowers and a good summary of this can be found in, for example, Takahashi [11], too. But, to state and prove our main theorem, we need some details on iterated generic ultrapowers, which we give here. For the purpose of this paper, we only need to treat iterations of length $\omega + 1$.

2.1 Fubini powers of ideals

Assume κ is a regular uncountable cardinal, I is a κ-complete ideal on κ and $n \in \omega$. Let I^n be the n-th Fubini power of I. (See below.) Then, as is the case with iterated ultrapowers, the n-th iterate of the generic ultrapower of V by I can be represented as a one-step generic ultrapower of V by I^n. We see this in the next subsection. Here we review basic properties of Fubini powers of ideals. Throughout this subsection, let κ and I be as above.

For each $n \in \omega$, the n-th Fubini power of I, I^n, is the ideal on ${}^n\kappa$ defined as follows: Let $I^0 := \{\emptyset\}$. Note that ${}^0\kappa = \{\langle\rangle\}$, where $\langle\rangle$ is the empty sequence. So I^0 is an ideal on ${}^0\kappa$ and $(I^0)^+ = (I^0)^* = \{\{\langle\rangle\}\}$. Assuming I^n was defined as an ideal on ${}^n\kappa$, let I^{n+1} be the ideal on ${}^{n+1}\kappa$ such that for each $A \subseteq {}^{n+1}\kappa$

$$A \in I^{n+1} \;\;\Leftrightarrow\;\; \{s \in {}^n\kappa \mid \; \{\xi < \kappa \mid s^\frown\langle\xi\rangle \in A\} \in I \;\; \} \;\in\; (I^n)^*.$$

It can be easily seen that I^{n+1} is a κ-complete ideal on ${}^{n+1}\kappa$. Note that I^1 and I are the same if we identify κ with ${}^1\kappa$ in the obvious way. (For each sequence s, $\langle\rangle^\frown s = s^\frown\langle\rangle = s$.)

The following lemma is basic:

Lemma 2.1. *Assume $m \leq n \in \omega$. Then for each $A \subseteq {}^n\kappa$*

(1) $A \in I^n \;\;\Leftrightarrow\;\; \{s \in {}^m\kappa \mid \; \{t \in {}^{n-m}\kappa \mid s^\frown t \in A\} \in I^{n-m} \;\} \;\in\; (I^m)^*$,

(2) $A \in (I^n)^+ \;\;\Leftrightarrow\;\; \{s \in {}^m\kappa \mid \; \{t \in {}^{n-m}\kappa \mid s^\frown t \in A\} \in (I^{n-m})^+ \;\} \;\in\; (I^m)^+$,

(3) $A \in (I^n)^* \;\;\Leftrightarrow\;\; \{s \in {}^m\kappa \mid \; \{t \in {}^{n-m}\kappa \mid s^\frown t \in A\} \in (I^{n-m})^* \;\} \;\in\; (I^m)^*$.

Proof. By induction on the lexicographical order of (n, m), we show (1)-(3) simultaneously. If $n = m = 0$ then (1)-(3) are trivial. Assume $m \leq n \in \omega$ and (1)-(3) are true for each pair m', n' such that $m' \leq n'$ and $(n', m') < (n, m)$. Because (2) and (3) follow from (1), it suffices to show (1) for m, n. If $m = n$ then (1) is trivial and if $m = n - 1$ then (1) is the definition of I^n. So we may assume $m < n - 1$.

Take an arbitrary $A \subseteq {}^n\kappa$. For each $s \in {}^{n-1}\kappa$, let A_s be $\{\xi < \kappa \mid s^\frown\langle\xi\rangle \in A\}$.

Then

$$
\begin{aligned}
& A \in I^n \\
\Leftrightarrow\ & \{s \in {}^{n-1}\kappa \mid A_s \in I\} \ \in\ (I^{n-1})^* \\
\Leftrightarrow\ & \{t \in {}^{m}\kappa \mid\ \{u \in {}^{n-1-m}\kappa \mid A_{t^\frown u} \in I\} \in (I^{n-1-m})^* \ \} \ \in\ (I^m)^* \\
\Leftrightarrow\ & \{t \in {}^{m}\kappa \mid\ \{v \in {}^{n-m}\kappa \mid t^\frown v \in A\} \in I^{n-m} \ \} \ \in\ (I^m)^*.
\end{aligned}
$$

The first and third equivalences follow from the definition of I^n and I^{n-m}. The second equivalence follows from the induction hypothesis. □

If $m \leq n < \omega$, there are a natural complete embedding and a projection between $\mathbb{P}_{I^m}$ and $\mathbb{P}_{I^n}$.

Let $\sigma_{m,n} : \mathcal{P}({}^m\kappa) \to \mathcal{P}({}^n\kappa)$ be the function such that for each $A \subseteq {}^m\kappa$,

$$\sigma_{m,n}(A) \ := \ \{s \in {}^n\kappa \mid s{\restriction}m \in A\}$$

and let $\pi_{n,m} : \mathcal{P}({}^n\kappa) \to \mathcal{P}({}^m\kappa)$ be the function such that for each $B \subseteq {}^n\kappa$,

$$\pi_{n,m}(B) \ := \ \{s \in {}^m\kappa \mid\ \{t \in {}^{n-m}\kappa \mid s^\frown t \in B\} \in (I^{n-m})^+ \ \}.$$

Note that if $m = n$ then $\sigma_{m,n} = \pi_{n,m} = id$.

By Lemma 2.1, if $m \leq n$ then $\sigma_{m,n}[(I^m)^+] \subseteq (I^n)^+$ and $\pi_{n,m}[(I^n)^+] \subseteq (I^m)^+$. Moreover, as we show in the next lemma, $\sigma_{m,n} \restriction (I^m)^+$ is a complete embedding from $\mathbb{P}_{I^m}$ to $\mathbb{P}_{I^n}$ and $\pi_{n,m} \restriction (I^n)^+$ is a projection from $\mathbb{P}_{I^n}$ to $\mathbb{P}_{I^m}$. We call $\sigma_{m,n}$ the natural complete embedding and call $\pi_{n,m}$ the natural projection associated with I.

Lemma 2.2. *Assume $l \leq m \leq n \in \omega$. Then the following hold:*

(1) $\sigma_{m,n} \circ \sigma_{l,m} = \sigma_{l,n}$.

(2) $\pi_{m,l} \circ \pi_{n,m} = \pi_{n,l}$.

(3) $\pi_{n,m} \circ \sigma_{m,n} = id \restriction \mathcal{P}({}^m\kappa)$.

(4) $A \in (I^m)^+ \Leftrightarrow \sigma_{m,n}(A) \in (I^n)^+$, *for each* $A \subseteq {}^m\kappa$.

(5) $A \in (I^n)^+ \Leftrightarrow \pi_{n,m}(A) \in (I^m)^+$, *for each* $A \subseteq {}^n\kappa$.

(6) $\sigma_{m,n} \restriction (I^m)^+ : \mathbb{P}_{I^m} \to \mathbb{P}_{I^n}$ *is a complete embedding.*

(7) $\pi_{n,m} \restriction (I^n)^+ : \mathbb{P}_{I^n} \to \mathbb{P}_{I^m}$ *is a good projection.*

Proof. (1) and (3) are clear by the definition of σ and π. (4) and (5) are clear by Lemma 2.1. So we prove (2), (6) and (7). We can assume $m < n$.

(2). Take an arbitrary $B \subseteq {}^n\kappa$. We show that $\pi_{m,l}(\pi_{n,m}(B)) = \pi_{n,l}(B)$. For each $s \in {}^l\kappa$, let $B_s := \{t \in {}^{n-l}\kappa \mid s^\frown t \in B\}$. Then, for each $s \in {}^l\kappa$,

$$\begin{aligned}
& s \in \pi_{m,l}(\pi_{n,m}(B)) \\
\Leftrightarrow\ & \{u \in {}^{m-l}\kappa \mid s^\frown u \in \pi_{n,m}(B)\} \in (I^{m-l})^+ \\
\Leftrightarrow\ & \{u \in {}^{m-l}\kappa \mid \ \{v \in {}^{n-m}\kappa \mid s^\frown u^\frown v \in B\} \in (I^{n-m})^+ \ \} \in (I^{m-l})^+ \\
\Leftrightarrow\ & \{u \in {}^{m-l}\kappa \mid \ \{v \in {}^{n-m}\kappa \mid u^\frown v \in B_s\} \in (I^{n-m})^+ \ \} \in (I^{m-l})^+ \\
\Leftrightarrow\ & B_s \in (I^{n-l})^+ \\
\Leftrightarrow\ & s \in \pi_{n,l}(B)\ .
\end{aligned}$$

(For the fourth equivalence, use (2) of Lemma 2.1.) Hence $\pi_{m,l}(\pi_{n,m}(B)) = \pi_{n,l}(B)$.

(6). Clearly $\sigma_{m,n}$ is order preserving and $A \perp B \ \rightarrow\ \sigma_{m,n}(A) \perp \sigma_{m,n}(B)$ for each $A, B \in \mathbb{P}_{(I^m)}$. So it suffices to show that if $M \subseteq \mathbb{P}_{I^m}$ is predense then $\sigma_{m,n}[M]$ is predense in $\mathbb{P}_{I^n}$. Assume $M \subseteq \mathbb{P}_{I^m}$ is predense. Take an arbitrary $A \in \mathbb{P}_{I^n}$. We must find $B \in M$ such that $\sigma_{m,n}(B) \cap A \in (I^n)^+$. Because $\pi_{n,m}(A) \in \mathbb{P}_{I^m}$ we can take $B \in M$ such that $B \cap \pi_{n,m}(A) \in (I^m)^+$. Then for each $s \in B \cap \pi_{n,m}(A)$,

$$\{t \in {}^{n-m}\kappa \mid s^\frown t \in \sigma_{m,n}(B) \cap A\} \ = \ \{t \in {}^{n-m}\kappa \mid s^\frown t \in A\} \ \in \ (I^{n-m})^+.$$

So, by Lemma 2.1, $\sigma_{m,n}(B) \cap A \in (I^n)^+$.

(7). Clearly $\pi_{n,m}$ is order preserving. By (3), $\pi_{n,m}$ is surjective. Assume $A \in \mathbb{P}_{I^n}$, $B \in \mathbb{P}_{I^m}$ and $B \leq \pi_{n,m}(A)$. Then, for each $s \in B$,

$$\{t \in {}^{n-m}\kappa \mid s^\frown t \in \sigma_{m,n}(B) \cap A\} \ = \ \{t \in {}^{n-m}\kappa \mid s^\frown t \in A\} \ \in \ (I^{n-m})^+.$$

So $C := \sigma_{m,n}(B) \cap A \in (I^n)^+$. Moreover, clearly, $C \leq A$ and $\pi_{n,m}(C) = B$. So $\pi_{n,m}$ is a projection. It is easy to see that $\pi_{n,m}$ is good. □

Lemma 2.3. *Assume I is normal. Let $n \in \omega$. Then $A \in (I^n)^*$ if and only if there is an $X \in I^*$ such that $A \subseteq [X]^n$, where $[X]^n$ is the set of all strictly increasing sequences of elements of X of length n.*

Proof. If $X \in I^*$ then it can be easily seen that $[X]^n \in (I^n)^*$. So $(\Leftarrow)$ is true. We show $(\Rightarrow)$ by induction on $n \in \omega$. If $n = 0$ or $n = 1$ then this is clear. So, assuming $n > 1$ and $(\Rightarrow)$ is true for $n-1$, we show this for n.

Assume $A \in (I^n)^*$. For each $t \in {}^{n-1}\kappa$, let $A_t := \{\xi < \kappa \mid t^\frown\langle\xi\rangle \in A\}$. Then $B := \{t \in {}^{n-1}\kappa \mid A_t \in I^*\} \in (I^{n-1})^*$. By the induction hypothesis, there is a $Y \in I^*$ such that $B \supseteq [Y]^{n-1}$. For each $\xi < \kappa$, let $A_\xi := \bigcap\{A_t \mid t \in B \wedge \max(t) < \xi\}$. Because I is κ-complete $A_\xi \in I^*$. Let $Z := \Delta_{\xi\in\kappa} A_\xi \in I^*$. Then let $X := Y \cap Z \cap \mathrm{Lim}(\kappa) \in I^*$. We show that if $s \in [X]^n$ then $s \in A$. Assume $s \in [X]^n$. Then, because $s \restriction n-1 \in [Y]^{n-1}$, $s \restriction n-1 \in B$. Let $\xi := s(n-2)+1$. Then $s(n-2) < \xi < s(n-1)$. Because $s(n-1) \in Z$, $s(n-1) \in A_\xi$ and so $s(n-1) \in A_{s\restriction n-1}$. This means $s \in A$. □

2.2 Iteration of generic ultrapowers

In this subsection, we review the iteration of generic ultrapowers of length at most $\omega + 1$.

All through this subsection, in V, fix κ, I, $\langle \mathbb{P}_n \mid n \in \omega \rangle$, $\langle \sigma_{m,n} \mid m \leq n < \omega \rangle$ and $\langle \pi_{n,m} \mid m \leq n < \omega \rangle$ so that

- κ is a regular uncountable cardinal,
- I is a normal precipitous ideal on κ,
- $\sigma_{m,n} : \mathcal{P}({}^m\kappa) \to \mathcal{P}({}^n\kappa)$ is the natural complete embedding,
- $\pi_{n,m} : \mathcal{P}({}^n\kappa) \to \mathcal{P}({}^m\kappa)$ is the natural projection associated with I,
- $\mathbb{P}_n := \mathbb{P}_{I^n}$.

Our first aim is to show:

- I^n is precipitous for each $n \in \omega$, i.e. if G is $(V, \mathbb{P}_{I^n})$-generic then $Ult(V, G)$ is well-founded.
- Assume G_n is $(V, \mathbb{P}_n)$-generic and, for each $m \leq n$, G_m is the $(V, \mathbb{P}_m)$-generic filter naturally obtained from G_n, i.e. $G_m = \pi_{n,m}[G_n] = \sigma_{m,n}^{-1}[G_n]$. In $V[G_n]$, let M_m be the transitive collapse of $Ult(V, G_m)$ for each $m \leq n$. Then $\langle M_m \mid m \leq n \rangle$ is an iteration of generic ultrapowers of V by I.

We begin with the factor lemma for $\mathbb{P}_n$.

Lemma 2.4. *Assume that $m, k \in \omega$, I^m is precipitous and G_m is a $(V, \mathbb{P}_m)$-generic filter. Let $j_m : V \to M_m \cong Ult(V, G_m)$ be the generic elementary embedding and κ_m, I_m, $\mathbb{P}_k^m$ be $j_m(\kappa)$, $j_m(I)$, $j_m(\mathbb{P}_k)$ respectively. Then, in $V[G_m]$, there is a surjective dense embedding from $\mathbb{P}_{m+k}/G_m$ to $\mathbb{P}_k^m$.*

Notation: Assume $m, k \in \omega$. For each $A \subseteq {}^{m+k}\kappa$ which is in V, let f_m^A be the function on ${}^m\kappa$ such that

$$f_m^A(s) \;=\; \{t \in {}^k\kappa \mid s^\frown t \in A\}$$

for each $s \in {}^m\kappa$. (Note that $f_m^A \in V$.)

Proof. First note that, in M_m, I_m is a normal ideal on κ_m and $\mathbb{P}_k^m = \mathbb{P}_{(I_m)^k}$.

In $V[G_m]$, let $d_k^m : \mathbb{P}_{m+k}/G_m \to \mathbb{P}_k^m$ be the function such that

$$d_k^m(A) \;:=\; [f_m^A]_{G_m}$$

for each $A \in \mathbb{P}_{m+k}$. Recall that the domain of $\mathbb{P}_{m+k}/G_m$ is $\pi_{m+k,m}^{-1}[G_m]$. (See "Notations and Facts" in Section 1.) So if $A \in \mathbb{P}_{m+k}/G_m$ then $d_k^m(A) \in \mathbb{P}_k^m$. Moreover it is clear that d_k^m is surjective and order preserving. So it suffices to show that, for each $A, B \in \mathbb{P}_{m+k}/G_m$, if $A \perp B$ in $\mathbb{P}_{m+k}/G_m$ then $d_k^m(A) \perp d_k^m(B)$ in $\mathbb{P}_k^m$.

Assume $d_k^m(A)$ and $d_k^m(B)$ are compatible. Then, by Łoś' theorem,

$$X := \{s \in {}^m\kappa \mid f_m^A(s) \cap f_m^B(s) \in (I^k)^+\} \in G_m .$$

So, by the definition of f_m^A and f_m^B, $\pi_{m+k,m}(A \cap B) = X \in G_m$. This means that $A \cap B \in \mathbb{P}_{m+k}/G_m$ and so A and B are compatible in $\mathbb{P}_{m+k}/G_m$. □

Next we show the factor lemma for a generic ultrapower of V by I^n. If G_{m+k} is $(V, \mathbb{P}_{m+k})$-generic and $G_m = \pi_{m+k,m}[G_{m+k}]$ then $d_k^m[G_{m+k}]$ is $(V[G_m], \mathbb{P}_k^m)$-generic, where $\mathbb{P}_k^m$ and d_k^m are as in the previous lemma. (Note that d_k^m is surjective.) Because $\mathbb{P}_k^m \in M_m \subseteq V[G_m]$, $d_k^m[G_{m+k}]$ is $(M_m, \mathbb{P}_k^m)$-generic. So, in $V[G_{m+k}]$, we can construct $Ult(M_m, d_k^m[G_{m+k}])$. We see that this model is isomorphic to $Ult(V, G_{m+k})$.

Lemma 2.5. *Assume $m, k \in \omega$, I^m is precipitous and G_{m+k} is $(V, \mathbb{P}_{m+k})$-generic. Let $G_m := \pi_{m+k,m}[G_{m+k}]$. In $V[G_m]$, let j_m, M_m, κ_m, I_m, $\mathbb{P}_k^m$ be as in Lemma 2.4 and let $d_k^m : \mathbb{P}_{m+k}/G_m \to \mathbb{P}_k^m$ be the dense embedding defined as in the proof of Lemma 2.4. In $V[G_{m+k}]$, let $G_k^m = d_k^m[G_{m+k}]$. Then $Ult(V, G_{m+k}) \cong Ult(M_m, G_k^m)$.*

Notation: Assume $m, k \in \omega$. For each function $g \in V$ on ${}^{m+k}\kappa$, let $f_m^g \in V$ be the function on ${}^m\kappa$ such that

$$f_m^g(s) = \text{the function on } {}^k\kappa \text{ such that } \forall t \in {}^k\kappa,\ f_m^g(s)(t) = g(s^\frown t)$$

for each $s \in {}^m\kappa$.

Proof. In $V[G_{m+k}]$, define $\tau : Ult(V, G_{m+k}) \to Ult(M_m, G_k^m)$ as

$$\tau((g)_{G_{m+k}}) := ([f_m^g]_{G_m})_{G_k^m}$$

for each $(g)_{G_{m+k}} \in Ult(V, G_{m+k})$. We show that τ is an isomorphism.

First we see that τ is well-defined, injective and elementary. Let $\varphi(v_1, ..., v_l)$ be a formula and $g_1, ..., g_l \in V$ be functions on ${}^{m+k}\kappa$. Then, by Łoś' theorem,

$$Ult(M_m, G_k^m) \models \varphi(([f_m^{g_1}]_{G_m})_{G_k^m}, \ldots, ([f_m^{g_l}]_{G_m})_{G_k^m}) \tag{1}$$

$$\Leftrightarrow \quad \{t \in {}^k\kappa_m \mid M_m \models \varphi([f_m^{g_1}]_{G^m}(t), \ldots, [f_m^{g_l}]_{G^m}(t))\} \in G_k^m . \tag{2}$$

Now, in V, let $A \subseteq {}^{m+k}\kappa$ be such that

$$A := \{u \in {}^{m+k}\kappa \mid V \models \varphi(g_1(u), ..., g_l(u))\}.$$

Then, for each $s \in {}^m\kappa$,

$$f_m^A(s) = \{t \in {}^k\kappa \mid V \models \varphi(f_m^{g_1}(s)(t), ..., f_m^{g_l}(s)(t))\} .$$

So, by Łoś' theorem,

$$[f_m^A]_{G_m} = \{t \in {}^k\kappa_m \mid M_m \models \varphi([f_m^{g_1}]_{G_m}(t), ..., [f_m^{g_l}]_{G_m}(t))\} .$$

Thus

$$\begin{array}{rcl} (1.2) & \Leftrightarrow & [f^A_m]_{G_m} \in G^m_k \\ & \Leftrightarrow & A \in G_{m+k} \\ & \Leftrightarrow & Ult(V, G_{m+k}) \vDash \varphi((g_1)_{G_{m+k}}, \ldots, (g_l)_{G_{m+k}}) \ . \end{array} \qquad (3)$$

For the second equivalence, recall that $d^m_k(A) = [f^A_m]_{G_m}$ and $G^m_k = d^m_k[G_{m+k}]$. The equivalence between (1.1) and (1.3) implies that τ is well-defined, injective and elementary. (For the well-definedness and injectivity, let φ be the formula "$v_1 = v_2$".)

Finally it is clear from the definition that τ is surjective. So τ is an isomorphism. □

Remark: If $Ult(V, G_{m+k})$ and $Ult(M_m, G^m_k)$ are well-founded then, because the above τ is an isomorphism,

$$[g]_{G_{m+k}} \ = \ [[f^g_m]_{G_m}]_{G^m_k}.$$

Lemma 2.6. *For each $m \in \omega$, I^m is precipitous.*

Proof. We show this by induction on $m \in \omega$. If $m = 1$, this is clear by the precipitousness of I. Assume I^m is precipitous. Assume G_{m+1} is $(V, \mathbb{P}_{m+1})$-generic. Let $G_m = \pi_{m+1,m}[G_{m+1}]$ and M_m, j_m, I_m, G^m_1 be as in Lemma 2.5. (Let $k = 1$.) Then G^m_1 is $(M_m, \mathbb{P}_{I_m})$-generic. On the other hand, by the elementarity of j_m, $M_m \vDash$ "I_m is precipitous". So $Ult(M_m, G^m_1)$ is well-founded. So, by Lemma 2.5, $Ult(V, G_{m+1})$ is well-founded. This shows I^{m+1} is precipitous. □

In the following lemma, note that if $m_1 \leq m_2 \leq n \in \omega$, G_n is $(V, \mathbb{P}_n)$-generic and $G_{m_j} = \pi_{n,m_j}[G_n]$ $(j = 1, 2)$ then $G_{m_1} = \pi_{m_2,m_1}[G_{m_2}]$.

Lemma 2.7. *Assume $n \in \omega$ and G_n is $(V, \mathbb{P}_n)$-generic. For each $m \leq n$, let $G_m := \pi_{n,m}[G_n]$ and M_m be the transitive collapse of $Ult(V, G_m)$. For each $m < n$, let G^m_1 be as in Lemma 2.5. Then $\langle M_m, G^l_1 \mid m \leq n,\ l < m\rangle$ is an iteration of generic ultrapowers of V by I.*

Proof. Clear by Lemma 2.5. □

In the rest of this subsection, we show basic facts needed in the next section. From now on, let W be an outer model of V in which there is a sequence $\langle G_n \mid n \in \omega\rangle$ such that if $m \leq n \in \omega$ then G_n is a $(V, \mathbb{P}_n)$-generic filter and $G_m = \pi_{n,m}[G_n]$. Basically we work in W. For each $m, k \in \omega$, let $j_m, M_m, \mathbb{P}^m_k$, etc. be as before, i.e.

- $j_m : V \to M_m \cong Ult(V, G_m)$ is the generic elementary embedding,
- $\kappa_m := j_m(\kappa)$, $I_m := j_m(I)$,
- $\mathbb{P}^m_k := j_m(\mathbb{P}_k) = (\mathbb{P}_{(I_m)^k})^{M_m}$,

- $d^m_k : \mathbb{P}_{m+k}/G_m \to \mathbb{P}^m_k$ is a dense embedding such that for each $A \in \mathbb{P}_{m+k}/G_m$,
$$d^m_k(A) := [f^A_m]_{G_m},$$
- $G^m_k := d^m_k[G_{m+k}]$.

First we give a representation for the map from M_m to M_n associated with the iteration of generic ultrapowers. For each $m \leq n \in \omega$, let $j_{m,n} : M_m \to M_n$ be the function defined as

$$j_{m,n}([g]_{G_m}) := [\bar{g}]_{G_n}$$

for each $[g]_{G_m} \in M_m$, where $\bar{g} \in V$ is the function on ${}^n\kappa$ such that $\bar{g}(s) = g(s \restriction m)$ for each $s \in {}^n\kappa$. It is easy to see that if $l \leq m \leq n \in \omega$ then $j_{0,n} = j_n$ and $j_{l,n} = j_{m,n} \circ j_{l,m}$.

Lemma 2.8. *Assume $m \leq n \in \omega$. Then $j_{m,n} : M_m \to M_n$ is the generic elementary embedding associated with $Ult(M_m, G^m_{n-m})$.*

Proof. Take an arbitrary $x \in M_m$ and assume $x = [g]_{G_m}$. We show that $j_{m,n}(x) = [c_x]_{G^m_{n-m}}$, where $c_x \in M_m$ is the constant function on ${}^{n-m}\kappa_m$ with the value x. Let $\bar{g}$ be as above, i.e. the function on ${}^n\kappa$ such that $\bar{g}(s) = g(s \restriction m)$ for each $s \in {}^n\kappa$. Then, in M_m, $[f^{\bar{g}}_m]_{G_m} = c_x$. By the Remark after Lemma 2.5, $[[f^{\bar{g}}_m]_{G_m}]_{G^m_{n-m}} = [\bar{g}]_{G_n}$. So

$$[c_x]_{G^m_{n-m}} = [[f^{\bar{g}}_m]_{G_m}]_{G^m_{m-n}} = [\bar{g}]_{G_n} = j_{m,n}(x) .$$

□

In particular, $j_{m,m+1} : M_m \to M_{m+1}$ is the ultrapower map associated with $Ult(V, G^m_1)$. Therefore $\langle M_n, G^m_1, j_{m,n} \mid m \leq n \in \omega \rangle$ is an iteration of generic ultrapowers of V by I.

Next we give the representation for the sequence of critical points. Because I is normal, the sequence of critical points has a good representation.

Lemma 2.9. *Assume $m < n \in \omega$. Then $\langle \kappa_k \mid m \leq k < n \rangle = [id \restriction {}^{n-m}\kappa_m]_{G^m_{n-m}}$. So, for each $A \subseteq {}^{n-m}\kappa_m$ which is in M_m, $A \in G^m_{n-m}$ if and only if $\langle \kappa_k \mid m \leq k < n \rangle \in j_{m,n}(A)$.*

Proof. For each $k < n$, let $i_k \in V$ be the function on ${}^n\kappa$ such that $i_k(s) = s(k)$ for each $s \in {}^n\kappa$. First we show $[i_k]_{G_n} = \kappa_k$. Let h_k be the function on ${}^{k+1}\kappa$ such that $h_k(s) = s(k)$ for each $s \in {}^{k+1}\kappa$. Then $j_{k+1,n}([h_k]_{G_{k+1}}) = [i_k]_{G_n}$. Because $j_{k+1,n}$ does not move κ_k, it suffices to show $[h_k]_{G_{k+1}} = \kappa_k$.

In V, $f^{h_k}_k(s) = id \restriction \kappa$ for each $s \in {}^k\kappa$. (Here we identified ${}^1\kappa$ with κ.) So, in M_k, $[f^{h_k}_k]_{G_k} = id \restriction \kappa_k$. Then, by the normality of I_k, $\kappa_k = [[f^{h_k}_k]_{G_k}]_{G^k_1}$. Then, by the remark after Lemma 2.5, $[h_k]_{G_{k+1}} = \kappa_k$.

Now let $g \in V$ be the function on ${}^{n}\kappa$ such that $g(s) = \langle s(m), s(m+1), ..., s(n-1)\rangle$ for each $s \in {}^{n}\kappa$. Then, because $f^g_m(s) = id \restriction {}^{n-m}\kappa$ for each $s \in {}^{m}\kappa$, $[f^g_m]_{G_m} = id \restriction {}^{n-m}\kappa_m$. So

$$[[f^g_m]_{G_m}]_{G^m_{n-m}} \;=\; [id \restriction {}^{n-m}\kappa_m]_{G^m_{n-m}}.$$

On the other hand,

$$[g]_{G_n} \;=\; \langle\, [i_m]_{G_n}\,,\, [i_{m+1}]_{G_n}\,, ...,\, [i_{n-1}]_{G_n}\,\rangle \;=\; \langle \kappa_m, \kappa_{m+1}, ..., \kappa_{n-1}\rangle.$$

So, by the remark after Lemma 2.5,

$$[id \restriction {}^{n-m}\kappa_m]_{G^m_{n-m}} \;=\; \langle \kappa_m, \kappa_{m+1}, ..., \kappa_{n-1}\rangle.$$

□

For each $m, k, l \in \omega$ such that $k \leq l$, let

- $\sigma^m_{k,l} \;:=\; j_m(\sigma_{k,l})$,
- $\pi^m_{l,k} \;:=\; j_m(\pi_{l,k})$.

Note that if m, k, l are as above then, in M_m,

- $\sigma^m_{k,l} : \mathcal{P}({}^{k}\kappa_m) \to \mathcal{P}({}^{l}\kappa_m)$ is the natural complete embedding,
- $\pi^m_{l,k} : \mathcal{P}({}^{l}\kappa_m) \to \mathcal{P}({}^{k}\kappa_m)$ is the natural projection associated with I_m.

Lemma 2.10. *Assume $m \in \omega$ and $k \leq l \in \omega$. Then the following diagrams commute. So $G^m_k = \pi^m_{l,k}[G^m_l] = (\sigma^m_{k,l})^{-1}[G^m_l]$.*

$$(1)\qquad \begin{array}{ccc} \mathbb{P}_{m+k}/G_m & \xrightarrow{\sigma_{m+k,m+l}} & \mathbb{P}_{m+l}/G_m \\ {\scriptstyle d^m_k}\big\downarrow & & \big\downarrow{\scriptstyle d^m_l} \\ \mathbb{P}^m_k & \xrightarrow{\sigma^m_{k,l}} & \mathbb{P}^m_l \end{array}$$

$$(2)\qquad \begin{array}{ccc} \mathbb{P}_{m+k}/G_m & \xleftarrow{\pi_{m+l,m+k}} & \mathbb{P}_{m+l}/G_m \\ {\scriptstyle d^m_k}\big\downarrow & & \big\downarrow{\scriptstyle d^m_l} \\ \mathbb{P}^m_k & \xleftarrow{\pi^m_{l,k}} & \mathbb{P}^m_l \end{array}$$

Proof.

(1): Assume $B \in \mathbb{P}_{m+k}/G_m$. Let $A := \sigma_{m+k,m+l}(B)$. Then $f^A_m(s) = \sigma_{k,l}(f^B_m(s))$ for each $s \in {}^{m}\kappa$. So, by Łoś' Theorem,

$$d^m_l(A) \;=\; [\,f^A_m\,]_{G_m} \;=\; \sigma^m_{k,l}([\,f^B_m\,]_{G_m}) \;=\; \sigma^m_{k,l}(d^m_k(B))\;.$$

(2): Assume $A \in \mathbb{P}_{m+l}/G_m$. Let $B := \pi_{m+l,m+k}(A)$. Then $f^B_m(s) = \pi_{l,k}(f^A_m(s))$ for each $s \in {}^{m}\kappa$. So

$$d^m_k(B) \;=\; [\,f^B_m\,]_{G_m} \;=\; \pi^m_{l,k}([\,f^A_m\,]_{G_m}) \;=\; \pi^m_{l,k}(d^m_l(A))\;.$$

□

We end this section with some notation. By Lemma 2.7, $\langle M_n, G_1^m, j_{m,n} \mid m \leq n < \omega\rangle$ is an iteration of generic ultrapowers of V by I. Then, by Fact 1.2, the direct limit of $\langle M_n, j_{m,n} \mid m \leq n < \omega\rangle$ is well-founded. Let M_ω be the transitive collapse of the direct limit of $\langle M_n, j_{m,n} \mid m \leq n \in \omega\rangle$ and, for each $m \in \omega$, let $j_{m,\omega} : M_m \to M_\omega$ be the induced elementary embedding. Then we call $\langle M_n, G_1^m, j_{m,n} \mid m \leq n \leq \omega,\ m < \omega\rangle$ *the iteration of generic ultrapowers of V by I associated with* $\langle G_n \mid n \in \omega\rangle$.

3 Generalized Prikry Forcing and Iteration of Generic Ultrapowers.

In this section, we generalize Theorem 1.1 for both PR^* and PR^+ (Theorem 3.3 and 3.5). After that we observe known facts about PR^* and PR^+ from the point of view of Theorem 3.3 and Theorem 3.5. Mathias found, in [8], a simple characterization for being a Prikry generic sequence and Theorem 1.1 is immediate from this characterization and Kunen's theory of iterated ultrapowers. But we do not know such simple characterizations for PR^* and PR^+, so we show Theorem 3.3 and 3.5 in a direct way.

3.1 PR^* and PR^+.

Before generalizing Theorem 1.1, we review basic properties of PR^* and PR^+.

First we give some definitions involving trees. Let α be an ordinal and $T \subseteq {}^{<\omega}\alpha$ be a tree. Then for each $t \in {}^{<\omega}\alpha$, let

- $t^\frown T := \bigcup\{t \restriction k \mid k < |t|\} \cup \bigcup\{t^\frown s \mid s \in T\}$,
- $T/t := \{s \in {}^{<\omega}\alpha \mid t^\frown s \in T\}$,
- $Suc_T(t) := \{\xi < \alpha \mid t^\frown\langle\xi\rangle \in T\}$,
- $[T] := \{b \in {}^{\omega}\alpha \mid \forall k \in \omega, b \restriction k \in T\}$ = the set of all infinite paths through T,
- $T_{(k)} := T \cap {}^{k}\alpha$ = the k-th level of T, for each $k \in \omega$.

Assume J is an ideal on some infinite ordinal α. For each tree $T \subset {}^{<\omega}\alpha$,

- T is called a J^*-*tree* if $T \neq \emptyset \wedge \forall t \in T,\ Suc_T(t) \in J^*$,
- T is called a J^+-*tree* if $T \neq \emptyset \wedge \forall t \in T,\ Suc_T(t) \in J^+$.

Let PR^*_J be the poset such that

$$PR^*_J \ = \ \{\langle t, T\rangle \mid t \in {}^{<\omega}\alpha \ \wedge \ T \subseteq {}^{<\omega}\alpha \text{ is a } J^*\text{-tree}\}$$

and, for each $\langle t_1, T_1\rangle, \langle t_2, T_2\rangle \in PR^*_J$, $\langle t_1, T_1\rangle \le \langle t_2, T_2\rangle$ iff $t_1{}^\frown T_1 \subseteq t_2{}^\frown T_2$. Let PR^+_J be the poset such that

$$PR^+_J \;=\; \{\langle t, T\rangle \mid t \in {}^{<\omega}\alpha \;\wedge\; T \subseteq {}^{<\omega}\alpha \text{ is a } J^+\text{-tree}\}$$

and, for each $\langle t_1, T_1\rangle, \langle t_2, T_2\rangle \in PR^+_J$, $\langle t_1, T_1\rangle \le \langle t_2, T_2\rangle$ iff $t_1{}^\frown T_1 \subseteq t_2{}^\frown T_2$.

The following lemma is basic.

Lemma 3.1. *Assume W is a transitive model of ZFC, $\alpha \in W$ is an infinite ordinal and $J \in W$ is such that $W \models$ "J is an ideal on α". Let $\mathbb{P}^* := (PR^*_J)^W$ and $\mathbb{P}^+ := (PR^+_J)^W$.*

(1) *Assume Γ is a $(W, \mathbb{P}^*)$-generic filter. Let $b := \bigcup\{t \mid \exists T, \langle t, T\rangle \in \Gamma\}$ and $\Gamma_b := \{\langle t, T\rangle \in \mathbb{P}^* \mid \forall n \in \omega, b{\restriction}n \in t{}^\frown T\}$. Then $\Gamma_b = \Gamma$.*

(2) *Assume Γ is a $(W, \mathbb{P}^+)$-generic filter. Let $b := \bigcup\{t \mid \exists T, \langle t, T\rangle \in \Gamma\}$ and $\Gamma_b := \{\langle t, T\rangle \in \mathbb{P}^+ \mid \forall n \in \omega, b{\restriction}n \in t{}^\frown T\}$. Then $\Gamma_b = \Gamma$.*

Proof. We show only (1). (2) can be shown in the same way. Clearly $\Gamma \subseteq \Gamma_b$. So it suffices to show that $\Gamma_b \subseteq \Gamma$.

Assume $\langle s, S\rangle \notin \Gamma$. Because

$$D \;:=\; \{\langle t, T\rangle \in \mathbb{P}^* \mid \langle t, T\rangle \le \langle s, S\rangle \;\text{ or }\; t \notin s{}^\frown S\,\}$$

is in W and is dense in $\mathbb{P}^*$, there is a $\langle t, T\rangle \in D \cap \Gamma$. Because $\langle s, S\rangle \notin \Gamma$, $t \notin s{}^\frown S$. Then, because t is an initial segment of b, $b \notin [s{}^\frown S]$. So $\langle s, S\rangle \notin \Gamma_b$. □

We call the above b's a PR^*_J-sequence or a PR^+_J-sequence. More precisely we make the following definitions.

Assume W is a transitive model of ZFC, $\alpha \in W$ is an infinite ordinal and $J \in W$ is such that $W \models$ "J is an ideal on α". Let $b \in {}^\omega\alpha$. Then we say:

- b is *a PR^*_J-sequence over W* if there is a $(W, (PR^*_J)^W)$-generic filter Γ such that $b = \bigcup\{t \mid \exists T, \langle t, T\rangle \in \Gamma\}$.
- b is *a PR^+_J-sequence over W* if there is a $(W, (PR^+_J)^W)$-generic filter Γ such that $b = \bigcup\{t \mid \exists T, \langle t, T\rangle \in \Gamma\}$.

By Lemma 3.1, b is a PR^*_J-sequence over W if and only if $\Gamma_b := \{\langle t, T\rangle \in (PR^*_J)^W \mid \forall n \in \omega, b \restriction n \in t{}^\frown T\}$ is a $(W, (PR^*_J)^W)$-generic filter. (For the backwards direction, note that if Γ_b is a generic filter then $b = \bigcup\{t \mid \exists T, \langle t, T\rangle \in \Gamma_b\}$.) This is also true for PR^+_J.

The following lemma is useful.

Lemma 3.2. *Assume W, α and J are as in Lemma 3.1 and $b, c \in {}^\omega\alpha$ have a common tail, i.e. $\exists m, n \in \omega\, \forall k \in \omega$, $b(m + k) = c(n + k)$. Then:*

(1) *b is a PR^*_J-sequence over W iff c is a PR^*_J-sequence over W.*

(2) *b is a PR^+_J-sequence over W iff c is a PR^+_J-sequence over W.*

Proof. We show only (1). Let W, α, J, b, c be as above. Let $m, n \in \omega$ be such that $\forall k \in \omega$, $b(m+k) = c(n+k)$ and let $u, v \in {}^{<\omega}\alpha$ be $b \restriction m$, $c \restriction n$ respectively.

Assume that b is a PR^*_J-sequence over W and Γ witnesses this. In W, let $\mathbb{P}_u$ be $PR^*_J \restriction \langle u, {}^{<\omega}\alpha\rangle$, i.e. the poset obtained from restricting PR^*_J to $\{\langle t, T\rangle \mid \langle t, T\rangle \leq \langle u, {}^{<\omega}\alpha\rangle\}$. Let $\mathbb{P}_v$ be $PR^*_J \restriction \langle v, {}^{<\omega}\alpha\rangle$. Then let $d : \mathbb{P}_u \to \mathbb{P}_v$ be such that

$$d(\langle u^\frown s, S\rangle) \ = \ \langle v^\frown s, S\rangle$$

for each $\langle u^\frown s, S\rangle \in \mathbb{P}_u$. Then $d \in W$ and d is an isomorphism. Because u is an initial segment of b, $\langle u, {}^{<\omega}\alpha\rangle \in \Gamma$. So $\Gamma \cap \mathbb{P}_u$ is $(W, \mathbb{P}_u)$-generic. So $d[\Gamma \cap \mathbb{P}_u]$ is $(W, \mathbb{P}_v)$-generic. So the filter Ω on $(PR^*_J)^W$ which is generated by $d[\Gamma \cap \mathbb{P}_u]$ is generic over W. Moreover,

$$\begin{aligned}\bigcup\{t \mid \exists T,\ \langle t, T\rangle \in \Omega\} &= \bigcup\{t \mid \exists T,\ \langle t, T\rangle \in d[\Gamma \cap \mathbb{P}_u]\}\\ &= \bigcup\{v^\frown s \mid \exists T, \langle u^\frown s, T\rangle \in \Gamma \cap \mathbb{P}_u\}\\ &= \bigcup\{v^\frown s \mid u^\frown s \in b\}\\ &= c\ .\end{aligned}$$

So Ω witnesses that c is a PR^*_J-sequence over W.

The other direction can be shown similarly. □

3.2 Generalizations of Theorem 1.1.

All through this subsection, in V, let κ be a regular uncountable cardinal and I be a normal precipitous ideal on κ. Moreover, for each $m \leq n \in \omega$, let $\mathbb{P}_n := \mathbb{P}_{I^n}$ and let $\sigma_{m,n} : \mathcal{P}({}^m\kappa) \to \mathcal{P}({}^n\kappa)$ and $\pi_{n,m} : \mathcal{P}({}^n\kappa) \to \mathcal{P}({}^m\kappa)$ be the natural complete embedding and the natural projection associated with I.

First we generalize Theorem 1.1 for PR^*.

Theorem 3.3. *Let $\mathbb{P}_\omega$ be the direct limit of $\langle \mathbb{P}_n, \sigma_{m,n} \mid m \leq n \in \omega\rangle$. Let G_ω be a $(V, \mathbb{P}_\omega)$-generic filter and, for each $n \in \omega$, let G_n be the $(V, \mathbb{P}_n)$-generic filter naturally obtained from G_ω. In $V[G_\omega]$, let $\langle M_n, H^m, j_{m,n} \mid m \leq n \leq \omega, m < \omega\rangle$ be the iteration of generic ultrapowers of V by I associated with $\langle G_n \mid n \in \omega\rangle$. Then $\langle j_{0,n}(\kappa) \mid n \in \omega\rangle$ is a $PR^*_{j_{0,\omega}(I)}$-sequence over M_ω.*

To prove the above theorem, we need some preliminaries. Until we complete the proof of the theorem, let $\mathbb{P}_\omega$, G_ω, $\langle G_n \mid n \in \omega\rangle$ and $\langle M_n, H^m, j_{m,n} \mid m \leq n \leq \omega, m < \omega\rangle$ be as in the theorem, and, in $V[G_\omega]$, let j_m, κ_m, I_m, $\mathbb{P}^m_k$, d^m_k, G^m_k, $\sigma^m_{k,l}$, $\pi^m_{l,k}$ be as in Section 2.2 for each $m, k, l \in \omega$ with $k \leq l$. Let $I_\omega := j_{0,\omega}(I)$. Note that $H^m = G^m_1$ for each $m \in \omega$.

$\mathbb{P}_\omega$ is the poset defined as follows: First let $\sim_\sigma$ be the equivalence relation on $\bigcup_{n\in\omega} \mathbb{P}_n$ such that for each $A, B \in \bigcup_{n\in\omega} \mathbb{P}_n$, say $A \in \mathbb{P}_m$ and $B \in \mathbb{P}_n$, $A \sim_\sigma B$ iff $\sigma_{m,l}(A) = \sigma_{n,l}(A)$, where $l = \max(m, n)$. Let $[A]_\sigma$ denote the equivalence class represented by A. Then

- $\mathbb{P}_\omega = \bigcup_{n\in\omega} \mathbb{P}_n / \sim_\sigma$,
- if $A \in \mathbb{P}_m$ and $B \in \mathbb{P}_n$ then, letting $l = \max(m, n)$, $[A]_\sigma \leq [B]_\sigma$ iff $\sigma_{m,l}(A) \leq \sigma_{n,l}(B)$ in $\mathbb{P}_l$.

For each $m \in \omega$, let $\sigma_{m,\omega} : \mathbb{P}_m \to \mathbb{P}_\omega$ be the complete embedding associated with the direct limit, i.e. the function such that $\sigma_{m,\omega}(A) = [A]_\sigma$ for each $A \in \mathbb{P}_m$. Then $G_m = \sigma_{m,\omega}^{-1}[G_\omega] = \{A \in \mathbb{P}_m \mid [A]_\sigma \in G_\omega\}$.

To prove the theorem, we need the factor lemma for $\mathbb{P}_\omega$ and $\langle M_n, H_m, j_{m,n} \mid m \leq n \leq \omega, m < \omega\rangle$. To see this we define $\mathbb{P}^m_\omega$, $\sigma^m_{k,\omega}$, d^m_ω and G^m_ω. Let $m \in \omega$.

Let

- $\mathbb{P}^m_\omega := j_m(\mathbb{P}_\omega)$,
- $\sigma^m_{k,\omega} := j_m(\sigma_{k,\omega})$, for each $k \in \omega$.

Recall that $\mathbb{P}^m_k = j_m(\mathbb{P}_k)$ and $\sigma^m_{k,l} = j_m(\sigma_{k,l})$ for each $k \leq l \in \omega$. So, in M_m, $\mathbb{P}^m_\omega$ is the direct limit of $\langle \mathbb{P}^m_k, \sigma^m_{k,l} \mid k \leq l \in \omega\rangle$ and $\sigma^m_{k,\omega} : \mathbb{P}^m_k \to \mathbb{P}^m_\omega$ is the induced complete embedding. For each $A \in \bigcup_{k\in\omega} \mathbb{P}^m_k$, let $[A]_{\sigma^m}$ denote the equivalence class represented by A. Then $\sigma^m_{k,\omega}(A) = [A]_{\sigma^m}$.

$d^m_\omega : \mathbb{P}_\omega/G_m \to \mathbb{P}^m_\omega$ is defined as follows. Note that $\mathbb{P}_\omega/G_m$ is the poset in $V[G_m]$ which is obtained from restricting $\mathbb{P}_\omega$ to $\{[A]_\sigma \mid \exists n \geq m,\ A \in \mathbb{P}_n/G_m\}$. (If $n \geq m$ and $A \in \mathbb{P}_n$ then $[A]_\sigma \in \mathbb{P}_\omega/G_m \Leftrightarrow \forall B \in G_m$, $[A]_\sigma$ and $[B]_\sigma$ are compatible in $\mathbb{P}_\omega \Leftrightarrow \forall B \in G_m$, A and $\sigma_{m,n}(B)$ are compatible in $\mathbb{P}_n \Leftrightarrow A \in \mathbb{P}_n/G_m$.) Then let $d^m_\omega : \mathbb{P}_\omega/G_m \to \mathbb{P}^m_\omega$ be the function such that

- $d^m_\omega([A]_\sigma) = [d^m_k(A)]_{\sigma^m}$,

for each $A \in \mathbb{P}_{m+k}/G_m$. By Lemma 2.10, d^m_ω is well-defined. Clearly $d^m_\omega \in V[G_m]$. We show that d^m_ω is a dense embedding.

Lemma 3.4. *$d^m_\omega : \mathbb{P}_\omega/G_m \to \mathbb{P}^m_\omega$ is a surjective dense embedding.*

Proof. Because $d^m_k : \mathbb{P}_{m+k}/G_m \to \mathbb{P}^m_k$ is surjective for each $k \in \omega$, d^m_ω is also surjective.

To see that d^m_ω is order preserving, assume $[A]_\sigma \leq [B]_\sigma$ in $\mathbb{P}_\omega/G_m$. Assume A, B are in $\mathbb{P}_{m+k}/G_m$, $\mathbb{P}_{m+l}/G_m$ respectively. Let $i := \max(k, l)$. Then $\sigma_{m+k,m+i}(A) \leq \sigma_{m+l,m+i}(B)$ in $\mathbb{P}_{m+i}/G_m$. Because d^m_i is order preserving, $d^m_i(\sigma_{m+k,m+i}(A)) \leq d^m_i(\sigma_{m+l,m+i}(B))$ in $\mathbb{P}^m_i$. Then $\sigma^m_{k,i}(d^m_k(A)) \leq \sigma^m_{l,i}(d^m_l(B))$ by Lemma 2.10. This means that $d^m_\omega([A]_\sigma) \leq d^m_\omega([B]_\sigma)$.

By replacing "$\leq$" by "$\perp$" in the above argument, we can see that d^m_ω preserves incompatibility. □

Let $G^m_\omega := d^m_\omega[G_\omega]$. Then G^m_ω is a $(V[G_m], \mathbb{P}^m_\omega)$-generic filter and so is $(M_m, \mathbb{P}^m_\omega)$-generic. We want to show that $G^m_k = (\sigma^m_{k,\omega})^{-1}(G^m_\omega)$ for each $k \in \omega$, i.e. G^m_k is the $(V[G_m], \mathbb{P}^m_k)$-generic filter naturally obtained from G^m_ω. Assume $k \in \omega$ and $A \in \mathbb{P}^m_k$. Let $B \in \mathbb{P}_{m+k}/G_m$ be such that $A = d^m_k(B)$. (Recall that d^m_k is surjective.) Then $d^m_\omega([B]_\sigma) = [A]_{\sigma^m}$. Then

$$[A]_{\sigma^m} \in G^m_\omega \Leftrightarrow [B]_\sigma \in G_\omega \Leftrightarrow B \in G_{m+k} \Leftrightarrow A \in G^m_k\,.$$

Thus $G^m_k = (\sigma^m_{k,\omega})^{-1}[G^m_\omega]$.

Note that, by Lemma 2.5, $\langle M_{m+l}, H^{m+k}, j_{m+k,m+l} \mid k \leq l \leq \omega, k < \omega\rangle$ is the iteration of generic ultrapowers of M_m by I_m naturally obtained from G^m_ω.

Now we can start to prove the theorem.

Proof of Theorem 3.3.

In $V[G_\omega]$, let $\vec{\kappa} := \langle \kappa_n \mid n \in \omega\rangle$ and let

$$\Gamma \;:=\; \{\langle t, T\rangle \in (PR^*_{I_\omega})^{M_\omega} \mid \forall n \in \omega, \vec{\kappa}\restriction n \in t^\frown T\}\;.$$

We show that Γ is a $(M_\omega, (PR^*_{I_\omega})^{M_\omega})$-generic filter. For simplicity of notation, we write $PR^*_{I_m}$ for $(PR^*_{I_m})^{M_m}$ for each $m \leq \omega$.

First we show the genericity of Γ. Let $D \in M_\omega$ be a dense subset of $PR^*_{I_\omega}$. We show $\Gamma \cap D \neq \emptyset$. Let $m \in \omega$ and $\bar{D} \in M_m$ be such that $D = j_{m,\omega}(\bar{D})$. $\bar{D}$ is a dense subset of $PR^*_{I_m}$. In M_m, define $E \subseteq \mathbb{P}^m_\omega$ as

$$E \;:=\; \{[A]_{\sigma^m} \in \mathbb{P}^m_\omega \mid \forall t \in A\, \exists T,\ \langle \vec{\kappa}\restriction m^\frown t,\ T\rangle \in \bar{D}\}\;.$$

Working in M_m, we show that E is dense in $\mathbb{P}^m_\omega$.

Claim 3.3.1. *Assume $k \in \omega$ and $s \in {}^k\kappa_m$. Then there is an $l \in \omega$ such that*

$$B^s_l \;:=\; \{t \in {}^l\kappa_m \mid \exists T,\ \langle \vec{\kappa}\restriction m^\frown s^\frown t,\ T\rangle \in \bar{D}\} \;\in\; ((I_m)^l)^+\;.$$

Proof of Claim. Let $k \in \omega$ and $s \in {}^k\kappa_m$. Assume $B^s_l \in (I_m)^l$ for every $l \in \omega$. Then, by Lemma 2.3, there is an $X_l \in (I_m)^*$ such that $[X_l]^l \cap B^s_l = \emptyset$ for each $l \in \omega$. Let $X := \bigcap_{l\in\omega} X_l$. Then $X \in (I_m)^*$ and if $t \in [X]^l$ then $t \notin B^s_l$. Then $[X]^{<\omega}$ is an $(I_m)^*$-tree and so $\langle \vec{\kappa}\restriction m^\frown s,\ [X]^{<\omega}\rangle \in PR^*_{I_m}$. But, by the construction of X, there is no element of $\bar{D}$ which extends $\langle \vec{\kappa}\restriction m^\frown s,\ [X]^{<\omega}\rangle$. This contradicts $\bar{D}$ is dense in $PR^*_{I_m}$. $\square$.*Claim*

Claim 3.3.2. *E is dense in $\mathbb{P}^m_\omega$.*

Proof of Claim. Let $k \in \omega$ and $A \in \mathbb{P}^m_k$. We find an element of E which extends $[A]_{\sigma^m}$. By the previous claim, for each $s \in A$, there is an $l_s \in \omega$ such that $B^s_{l_s}$ is $(I_m)^{l_s}$-positive. Because $(I_m)^k$ is κ_m-complete, there is an $A' \subseteq A$ and $l \in \omega$ such that A' is $(I_m)^k$-positive and $l_s = l$ for every $s \in A'$. Then let $B := \{s^\frown t \mid s \in A' \wedge t \in B^s_l\}$. Because B^s_l is $(I_m)^l$-positive for each $s \in A'$, B is $(I_m)^{k+l}$-positive, i.e. $B \in \mathbb{P}^m_{k+l}$. Then clearly $\sigma^m_{k,k+l}(A) \geq B$ in $\mathbb{P}^m_{k+l}$ and so $[A]_{\sigma^m} \geq [B]_{\sigma^m}$. On the other hand, if $u \in B$ then there is a T such that $\langle \vec{\kappa}\restriction m^\frown u,\ T\rangle \in \bar{D}$. So $[B]_{\sigma^m} \in E$. $\square$.*Claim*

Return to $V[G_\omega]$.

Because G^m_ω is $(M_m, \mathbb{P}^m_\omega)$-generic, $G^m_\omega \cap E \neq \emptyset$. Let A be such that $[A]_{\sigma^m} \in G^m_\omega \cap E$ and A witnesses that $[A]_{\sigma^m} \in E$, i.e. $\forall t \in A\, \exists T,\ \langle \vec{\kappa}\restriction m^\frown t,\ T\rangle \in \bar{D}$. Assume $A \in \mathbb{P}^m_k$. Then $A \in G^m_k$ and so, by Lemma 2.9, $\vec{\kappa}\restriction [m, m+k) \in j_{m,m+k}(A)$. On the other hand, because $j_{m,m+k}$ is an elementary embedding and does not move $\vec{\kappa}\restriction m$,

$$M_{m+k} \;\models\; \forall t \in j_{m,m+k}(A)\, \exists T,\ \langle \vec{\kappa}\restriction m^\frown t,\ T\rangle \in j_{m,m+k}(\bar{D})\;.$$

So, in M_{m+k}, there exists an $(I_{m+k})^*$-tree $\bar{T}$ such that $\langle \vec{\kappa} \restriction m+k, \bar{T}\rangle \in j_{m,m+k}(\bar{D})$. Let $T := j_{m+k,\omega}(\bar{T})$. Then

$$\langle \vec{\kappa}\restriction m+k,\, T\rangle \;=\; j_{m+k,\omega}(\langle \vec{\kappa}\restriction m+k, \bar{T}\rangle) \;\in\; j_{m+k,\omega}(j_{m,m+k}(\bar{D})) \;=\; D\ .$$

Thus it suffices to show that $\langle \vec{\kappa}\restriction m+k,\, T\rangle \in \Gamma$. To see this it suffices to show that, for every $l > 0$, $\langle \kappa_{m+k}, ..., \kappa_{m+k+l-1}\rangle \in T$. Assume $l > 0$. Let $n := m+k$. Because $\bar{T}$ is an $(I_n)^*$-tree the l-th level of $\bar{T}$, $\bar{T}_{(l)}$, is in $((I_n)^l)^*$. So $\bar{T}_{(l)} \in G^n_l$. Then, by Lemma 2.9, $\langle \kappa_n, ..., \kappa_{n+l-1}\rangle \in j_{n,n+l}(\bar{T}_{(l)})$. Then,

$$\langle \kappa_n, ..., \kappa_{n+l-1}\rangle \;=\; j_{n+l,\omega}(\langle \kappa_n, ..., \kappa_{n+l-1}\rangle) \;\in\; j_{n+l,\omega}(j_{n,n+l}(\bar{T}_{(l)})) \;=\; T_{(l)}\ .$$

This completes the proof of the genericity.

Next we show that Γ is a filter. Clearly Γ is closed upwards. We show that if $\langle t_1, T_1\rangle$ and $\langle t_2, T_2\rangle$ are in Γ then they are compatible in $PR^*_{I_\omega}$. (Because of the genericity of Γ, this suffices.) Assume $\langle t_1, T_1\rangle$, $\langle t_2, T_2\rangle$ are in Γ. Let $n \in \omega$ be such that $t_1, t_2 \subseteq \vec{\kappa}\restriction n$. Then let

$$S_i \;:=\; (t_i{}^\frown T_i)\,/\,(\vec{\kappa}\restriction n)$$

for $i = 0, 1$. Because $\vec{\kappa}\restriction n \in t_i{}^\frown T_i$, S_i is an $(I_\omega)^*$-tree. Then $\langle \vec{\kappa}\restriction n,\, S_1 \cap S_2\rangle$ is in $PR^*_{I_\omega}$ and is a common extension of $\langle t_1, T_1\rangle$ and $\langle t_2, T_2\rangle$.

This completes the proof of Theorem 3.3. $\square$. *Theorem*

Next we generalize Theorem 1.1 for PR^+.

Theorem 3.5. *Let $\mathbb{P}_\omega$ be the inverse limit of $\langle \mathbb{P}_n, \pi_{n,m} \mid m \leq n \in \omega\rangle$. Let G_ω be a $(V, \mathbb{P}_\omega)$-generic filter and, for each $n \in \omega$, let G_n be a $(V, \mathbb{P}_n)$-generic filter naturally obtained from G_ω. In $V[G_\omega]$, let $\langle M_n, H^m, j_{m,n} \mid m \leq n \leq \omega, m < \omega\rangle$ be the iteration of generic ultrapowers of V by I associated with $\langle G_n \mid n \in \omega\rangle$. Then $\langle j_{0,n}(\kappa) \mid n \in \omega\rangle$ is a $PR^+_{j_{0,\omega}(I)}$-sequence over M_ω.*

To prove the theorem we need some preliminaries. Until we complete the proof of the above theorem, let $\mathbb{P}_\omega$, G_ω, $\langle G_n \mid n \in \omega\rangle$, $\langle M_n, H^m, j_{m,n} \mid m \leq n \leq \omega, m < \omega\rangle$ be as in the theorem. In $V[G_\omega]$, let j_m, κ_m, I_m, $\mathbb{P}^m_k$, d^m_k, G^m_k, $\sigma^m_{k,l}$, $\pi^m_{l,k}$ be as in Section 2.2 for each $m, k, l \in \omega$ with $k \leq l$. Let $I_\omega = j_{0,\omega}(I)$. Note that $H^m = G^m_1$.

$\mathbb{P}_\omega$ is the poset such that

- $\mathbb{P}_\omega$ is the set of all sequences $\langle A_n \mid n \in \omega\rangle$ such that $\pi_{n,m}(A_n) = A_m$ for each $m \leq n \in \omega$,
- $\langle A_n \mid n \in \omega\rangle \leq \langle B_n \mid n \in \omega\rangle$ iff $A_n \leq B_n$ in $\mathbb{P}_n$ for every $n \in \omega$.

First we modify $\mathbb{P}_\omega$. In V, let $\mathbb{P}$ be the poset of all I^+-trees ordered by inclusion. We see that $\mathbb{P}_\omega$ and $\mathbb{P}$ are equivalent. Note that if T is an I^+-tree then the sequence of levels of T, $\langle T_{(n)} \mid n \in \omega\rangle$, is in $\mathbb{P}_\omega$. Let $e : \mathbb{P} \to \mathbb{P}_\omega$ be the function defined by $e(T) := \langle T_{(n)} \mid n \in \omega\rangle$.

Lemma 3.6. *e is a dense embedding.*

Proof. Clearly e is order preserving. Moreover, if $e(T_1) \leq e(T_2)$ in $\mathbb{P}_\omega$ then $T_1 \leq T_2$ in $\mathbb{P}$. So it suffices to show that $e[\mathbb{P}]$ is dense in $\mathbb{P}_\omega$.

Take an arbitrary $\langle B_n \mid n \in \omega\rangle \in \mathbb{P}_\omega$. By induction on $n \in \omega$, define $A_n \subseteq B_n$ as follows. Let $A_0 := B_0 = \{\langle\rangle\}$. Assuming $A_n \subseteq B_n$ is defined, let $A_{n+1} := \{s \in B_{n+1} \mid s \restriction n \in A_n\}$. Then $T := \bigcup_{n\in\omega} A_n$ is a tree. Moreover, because $A_n \subseteq B_n = \pi_{n+1,n}(B_{n+1})$, $\{\xi \in \kappa \mid s^\frown\langle\xi\rangle \in A_{n+1}\} \in I^+$ for each $s \in A_n$. Thus T is an I^+-tree. Hence $e(T) = \langle A_n \mid n \in \omega\rangle \leq \langle B_n \mid n \in \omega\rangle$. □

We argue using $\mathbb{P}$ instead of $\mathbb{P}_\omega$. Let $G := e^{-1}[G_\omega]$. Then G is $(V, \mathbb{P})$-generic. In V, let $\pi_n : \mathbb{P} \to \mathbb{P}_n$ be the function defined by $\pi_n(T) := T_{(n)}$ for each $T \in \mathbb{P}$. Then π_n is the composition of e and the natural projection from $\mathbb{P}_\omega$ to $\mathbb{P}_n$. Thus G_n is the filter generated by $\pi_n[G] = \{T_{(n)} \mid T \in G\}$.

As in Theorem 3.3, we need the factor lemma for $\mathbb{P}$ and $\langle M_n, H^m, j_{m,n} \mid m \leq n \leq \omega, m < \omega\rangle$. We define $\mathbb{P}^m$, π^m_k, d^m and G^m. Let $m \in \omega$.

Let

- $\mathbb{P}^m := j_m(\mathbb{P})$,
- $\pi^m_k := j_m(\pi_k)$.

In M_m, $\mathbb{P}^m$ is the poset of all $(I_m)^+$-trees ordered by inclusion and π^m_k is the function defined by $\pi^m_k(T) := T_{(k)}$.

$d^m : \mathbb{P}/G_m \to \mathbb{P}^m$ is defined similarly to d^m_k. For each $T \in \mathbb{P}$, let $f^T_m \in V$ be the function on ${}^m\kappa$ such that $f^T_m(t) = T/t$ for each $t \in {}^m\kappa$. Note that $f^T_m(t)$ is an I^+-tree for each $t \in T_{(m)}$. So if $T \in \mathbb{P}/G_m$, i.e. $T_{(m)} \in G_m$ then $[f^T_m]_{G_m} \in \mathbb{P}^m$. In $V[G_m]$, define $d^m : \mathbb{P}/G_m \to \mathbb{P}^m$ by $d^m(T) := [f^T_m]_{G_m}$ for each $T \in \mathbb{P}/G_m$.

Lemma 3.7. *d^m is a surjective dense embedding.*

Proof. This can be shown in the same way as Lemma 2.4. We show only that d^m preserves incompatibility.

Assume that $T_1, T_2 \in \mathbb{P}/G_m$ and $d^m(T_1), d^m(T_2)$ are compatible in $\mathbb{P}^m$. We show that T_1, T_2 are compatible in $\mathbb{P}/G_m$. Let $g \in V$ be such that $[g]_{G_m}$ is a common extension of $d^m(T_1)$ and $d^m(T_2)$. We may assume $g(t)$ is an I^+-tree for each $t \in {}^m\kappa$. Because $[g]_{G_m}$ is a common extension, $B := \{t \in {}^m\kappa \mid g(t) \subseteq T_1/t\,,\, T_2/t\} \cap T_{1(m)} \cap T_{2(m)} \in G_m$. Because $\pi_m[\mathbb{P}]$ is dense in $\mathbb{P}_m$, there is an $A \subseteq B$ such that $A \in G_m$ and $A \in \pi_m[\mathbb{P}]$. Then A is the m-th level of some I^+-tree and $g(t)$ is an I^+-tree for each $t \in A$. So $T := \bigcup\{t^\frown g(t) \mid t \in A\}$ is an I^+-tree. Moreover $T \subseteq T_1, T_2$ and $T \in \mathbb{P}/G_m$. Thus T_1 and T_2 are compatible in $\mathbb{P}/G_m$. □

Let $G^m := d^m[G]$. Then G^m is $(V[G_m], \mathbb{P}^m)$-generic and thus $(M_m, \mathbb{P}^m)$-generic. We show that each G^m_k is the $\mathbb{P}^m_k$-generic filter naturally obtained from G^m.

Lemma 3.8. *Assume $k \in \omega$. Then G^m_k is the filter generated by $\pi^m_k[G^m] = \{T_{(k)} \mid T \in G^m\}$.*

Proof. G^m_k is an $(M_m, \mathbb{P}^m_k)$-generic filter and $\pi^m_k[G^m]$ generates an $(M_m, \mathbb{P}^m_k)$-generic filter. So it suffices to show that $\pi^m_k[G^m] \subseteq G^m_k$. For each $A \in \mathbb{P}_{m+k}$, let $f^A_m \in V$ be as in Lemma 2.4. Recall that $d^m_k(A) = [f^A_m]_{G_m}$ for each $A \in \mathbb{P}_{m+k}/G_m$ and that $G^m_k = d^m_k[G_{m+k}]$.

Take an arbitrary $B \in \pi^m_k[G^m]$. Then there is an $S \in G^m$ such that $S_{(k)} = B$. Let $T \in G$ be such that $d^m(T) = S$ and $A := T_{(m+k)}$. Note that $A \in G_{m+k}$. Then, for each $t \in T_{(m)}$, $f^A_m(t)$ is the k-th level of $f^T_m(t)$. So, in M_m, $[f^A_m]_{G_m}$ is the k-th level of $[f^T_m(t)]_{G_m}$. Thus $d^m_k(A) = S_{(k)} = B$. Because $A \in G_{m+k}$, $B \in G^m_k$. □

Note that $\langle M_{m+l}, H^{m+k}, j_{m+k,m+l} \mid k \leq l \leq \omega, k < \omega\rangle$ is the iteration of generic ultrapowers of M_m by I_m naturally obtained from G^m.

Now we can start to prove the theorem.

Proof of Theorem 3.5.

In $V[G]$, let $\vec{\kappa} := \langle \kappa_n \mid n \in \omega\rangle$ and let

$$\Gamma \ := \ \{\langle t, T\rangle \in (PR^+_{I_\omega})^{M_\omega} \mid \forall n \in \omega, \vec{\kappa}\restriction n \in t^\frown T\}.$$

We show that Γ is $(M_\omega, (PR^+_{I_\omega})^{M_\omega})$-generic. For simplicity of notation, we write $PR^+_{I_n}$ for $(PR^+_{I_n})^{M_n}$ for each $n \leq \omega$.

First we show the genericity. Let $D \in M_\omega$ be a dense subset of $PR^+_{I_\omega}$. We show that $\Gamma \cap D \neq \emptyset$. There is an $m \in \omega$ and $\bar{D} \in M_m$ such that $j_{m,\omega}(\bar{D}) = D$. Then $\bar{D}$ is dense in $PR^+_{I_m}$. In M_m, let $E \subseteq \mathbb{P}^m$ be defined by

$$E \ = \ \{T \in \mathbb{P}^m \mid \exists k \in \omega \forall t \in T_{(k)},\ \langle \vec{\kappa}\restriction m^\frown t,\ T/t\rangle \in \bar{D}\}.$$

Working in M_m, we show that E is dense in $\mathbb{P}^m$. Take an arbitrary $S \in \mathbb{P}^m$. We find a $T \in E$ such that $T \leq S$.

Claim 3.5.1. *For some $k \in \omega$,*

$$B_k \ := \ \{s \in S_{(k)} \mid \exists S', \langle \vec{\kappa}\restriction m, S\rangle \geq \langle \vec{\kappa}\restriction m^\frown s, S'\rangle \in \bar{D}\} \ \in \ ((I_m)^k)^+.$$

Proof of Claim. Assume not. Then, for each $k \in \omega$, there is an $X_k \in (I_m)^*$ such that $[X_k]^k \cap B_k = \emptyset$. Let $X := \bigcap_{k\in\omega} X_k$. Then $X \in (I_m)^*$ and so $S \cap [X]^{<\omega}$ is an $(I_m)^+$-tree. Thus $\langle \vec{\kappa}\restriction m, S \cap [X]^{<\omega}\rangle \in PR^+_{I_m}$. But if $s \in S \cap [X]^{<\omega}$ then $s \notin B_{|s|}$. Hence there is no element of $\bar{D}$ which extends $\langle \vec{\kappa}\restriction m, S \cap [X]^{<\omega}\rangle$. This contradicts $\bar{D}$ is dense in $PR^+_{I_m}$. □.*Claim*

Let $k \in \omega$ be such that B_k is $(I_m)^k$-positive. For each $s \in B_k$, let S_s be an $(I_m)^+$-tree witnessing $s \in B_k$. Note that $s^\frown S_s \subseteq S$. Because $\pi^m_k[\mathbb{P}^m]$ is dense in $\mathbb{P}^m_k$, there is an $A \subseteq B_k$ such that A is the k-th level of some $(I_m)^+$-tree. Then

$$T \ := \ \bigcup\{s^\frown S_s \mid s \in A\}$$

is an $(I_m)^+$-tree. Moreover $T \subseteq S$ and k witnesses that $T \in E$. This shows that E is dense.

Return to $V[G]$.

Let $\bar{T}$ be in $G^m \cap E$ and let k be the element of ω witnessing that $\bar{T} \in E$. Let $T := j_{m,\omega}(\bar{T})$.

Claim 3.5.2. *For each $l \in \omega$, $\vec{\kappa} \restriction [m, m+l) \in T_{(l)}$.*

Proof of Claim. Let $l \in \omega$. Because $\bar{T} \in G^m$, $\bar{T}_{(l)} \in G^m_l$ by Lemma 3.8. Thus, by Lemma 2.9, $\vec{\kappa} \restriction [m, m+l) \in j_{m,m+l}(\bar{T}_{(l)})$. Then, because $j_{m+l,\omega}$ does not move κ_{m+i} for each $i < l$, $\vec{\kappa} \restriction [m, m+l) \in j_{m+l,\omega}(j_{m,m+l}(\bar{T}_{(l)})) = T_{(l)}$. $\square$.*Claim*

Because $\bar{T} \in E$ and k witnesses this,

$$M_m \ \models \ \text{“}\forall t \in \bar{T}_{(k)},\ \langle \vec{\kappa} \restriction m^\frown t,\ \bar{T}/t \rangle \in \bar{D}\text{”}.$$

Thus, because $j_{m,\omega}$ is elementary and $\vec{\kappa} \restriction [m, m+k) \in T_{(k)}$,

$$\langle \vec{\kappa} \restriction m^\frown \vec{\kappa} \restriction [m, m+k),\ T/\vec{\kappa} \restriction [m, m+k) \rangle \ = \ \langle \vec{\kappa} \restriction m+k,\ T/\vec{\kappa} \restriction [m, m+k) \rangle \ \in \ D.$$

On the other hand, by the previous claim,

$$\langle \vec{\kappa} \restriction m+k,\ T/\vec{\kappa} \restriction [m, m+k) \rangle \in \Gamma.$$

So $\Gamma \cap D \neq \emptyset$.

Next we show that Γ is a filter. Clearly Γ is closed upwards. So, because of the genericity, it suffices to show that if $\langle t_1, T_1 \rangle$, $\langle t_2, T_2 \rangle \in \Gamma$ then $\langle t_1, T_1 \rangle$ and $\langle t_2, T_2 \rangle$ are compatible in $PR^+_{I_\omega}$.

Assume $\langle t_1, T_1 \rangle, \langle t_2, T_2 \rangle \in \Gamma$ and $\langle t_1, T_1 \rangle \perp \langle t_2, T_2 \rangle$ in $PR^+_{I_\omega}$. Let

$$D \ := \ \{ \langle t, T \rangle \in PR^+_{I_\omega} \mid t \notin t_1{}^\frown T_1 \ \vee \ t \notin t_2{}^\frown T_2 \}.$$

Then D is dense in $PR^+_{I_\omega}$. Let $\langle t, T \rangle \in \Gamma \cap D$. Without loss of generality, we may assume $t \notin t_1{}^\frown T_1$. Then, because $\langle t, T \rangle \in \Gamma$, t is an initial segment of $\vec{\kappa}$. Therefore $\vec{\kappa} \restriction |t| = t \notin t_1{}^\frown T_1$. This contradicts that $\langle t_1, T_1 \rangle \in \Gamma$.

This completes the proof of Theorem 3.5. $\square$.*Theorem*

3.3 Observations about PR^* and PR^+ from Theorem 3.3 and 3.5.

If κ is a measurable cardinal and U is a normal measure on κ then the Prikry forcing associated with U does not affect V_κ. It is known that this can be generalized for PR^* and PR^+. In this subsection, we observe this from the point of view of Theorem 3.3 and 3.5.

First we define strategic closure of ideals.

For each poset $\mathbb{Q}$ and $\delta \in On$, let $\Omega_\delta(\mathbb{Q})$ be the following two player game of length δ. In $\Omega_\delta(\mathbb{Q})$, Player I and II in turn choose an element of $\mathbb{Q}$ and build a descending chain in $\mathbb{Q}$, $q_1 \geq q_2 \geq q_3 \geq ... \geq q_\xi \geq q_{\xi+1} \geq ...$. Player I plays q_ξ for odd ξ's and II plays for even and limit ξ's. Player II wins if and only if the game can be continued to build a descending chain $\langle q_\xi \mid \xi \in \delta - \{0\}\rangle$. Otherwise I wins.

We say that $\mathbb{Q}$ is *δ-strategically closed* if Player II has a winning strategy in the game $\Omega_\delta(\mathbb{Q})$. Here, a winning strategy for Player II is a function τ from the set of all initial plays of $\Omega_\delta(\mathbb{Q})$ to $\mathbb{Q}$ such that if Player II plays $\tau(\langle q_\eta \mid \eta \in \xi - \{0\}\rangle)$ in each ξ-th stage then Player II wins. An ideal I is called *δ-strategically closed* if $\mathbb{P}_I$ is δ-strategically closed.

Note that every $\omega + 1$-strategically closed ideal is precipitous. In Galvin-Jech-Magidor [4], it is shown that if κ is measurable and J is a normal maximal ideal on κ then $\bar{J}$ is $\omega + 1$-strategically closed in $V^{Col(\omega_1,<\kappa)}$, where $\bar{J}$ is the ideal generated by J. In general, if κ and J are as above and $\gamma < \kappa$ is a regular uncountable cardinal then $\bar{J}$ is γ-strategically closed in $V^{Col(\gamma,<\kappa)}$.

Lemma 3.9. *Let κ be a regular uncountable cardinal and I be a normal ideal on κ. For each $m \leq n \in \omega$, let $\sigma_{m,n} : \mathbb{P}_{I^m} \to \mathbb{P}_{I^n}$ and $\pi_{n,m} : \mathbb{P}_{I^n} \to \mathbb{P}_{I^m}$ be the natural complete embedding and the natural projection associated with I, respectively.*

(1) *Assume $\delta < \kappa$ and I is δ-saturated. Then the direct limit of $\langle \mathbb{P}_{I^n}, \sigma_{m,n} \mid m \leq n \in \omega\rangle$ has the δ-c.c.*

(2) *Assume $\delta > \omega$ and I is δ-strategically closed. Then the inverse limit of $\langle \mathbb{P}_{I^n}, \pi_{n,m} \mid m \leq n \in \omega\rangle$ is δ-strategically closed.*

Proof. For each $n \in \omega$, let $\mathbb{P}_n := \mathbb{P}_{I^n}$ and $\dot{G}_n$ be the canonical name of a $\mathbb{P}_n$-generic filter.
(1). It suffices to show that $\Vdash_{\mathbb{P}_n}$ "$\mathbb{P}_{n+1}/\dot{G}_n$ has the δ-c.c.". We show this by induction on $n \in \omega$. Note that if $n = 1$ then this is true because I is δ-saturated. Assume $n \in \omega$ and that this is true for each $m \leq n$. We show this for $n + 1$.

Let G_n be a $(V, \mathbb{P}_n)$-generic filter and let $j_n : V \to M_n \cong Ult(V, G_n)$ be the generic elementary embedding. Because j_n is elementary and j_n does not move δ, $j_n(\mathbb{P}_I)$ has δ-c.c. in M_n. By the induction hypothesis, in V, $\mathbb{P}_n$ has the δ-c.c. and so I^n is a κ-complete δ-saturated ideal. Thus ${}^\kappa M_n \cap V[G_n] \subseteq M_n$. So $j_n(\mathbb{P}_I)$ has the δ-c.c. in $V[G_n]$. By Lemma 2.4, $\mathbb{P}_{n+1}/G_n$ and $j_n(\mathbb{P}_I)$ are equivalent in $V[G_n]$. Thus $\mathbb{P}_{n+1}/G_n$ has the δ-c.c. in $V[G_n]$.
(2). (2) can be shown in the same way as (1). But we need a slightly long argument to treat the inverse limit of posets. So here we directly prove that $\mathbb{P}$ in the proof of Theorem 3.5 is δ-strategically closed. Recall that $\mathbb{P}$ is the poset of all I^+-trees ordered by inclusion and $\mathbb{P}$ is forcing equivalent to the inverse limit of $\langle \mathbb{P}_{I^n}, \pi_{n,m} \mid m \leq n \in \omega\rangle$. Let τ be a winning strategy for Player II in the game $\Omega_\delta(\mathbb{P}_I)$. Using τ, we give a winning strategy $\bar{\tau}$ for Player II in $\Omega_\delta(\mathbb{P})$.

Let ξ be in $\delta - \{0\}$ and $\langle T_\eta \mid \eta \in \xi - \{0\}\rangle$ be a descending sequence in $\mathbb{P}$. Let $S := \bigcap_{\eta \in \xi - \{0\}} T_\eta$ and let

$$T \ := \ \{t \in S \mid \forall k \in |t|,\ t(k) \in \tau(\langle Suc_{T_\eta}(t \restriction k) \mid \eta \in \xi - \{0\}\rangle)\}\,.$$

Note that T is a tree. Moreover if $t \in T$ and $\langle Suc_{T_\eta}(t) \mid \eta \in \xi - \{0\}\rangle$ is an initial play of $\Omega_\delta(\mathbb{P}_I)$ in which II has played according to τ then $Suc_T(t) =$

$\tau(\langle Suc_{T_\eta}(t) \mid \eta \in \xi - \{0\}\rangle)$. If $T \in \mathbb{P}$ then let $\bar\tau(\langle T_\eta \mid \eta \in \xi - \{0\}\rangle)$ be the above T. Otherwise let $\bar\tau(\langle T_\eta \mid \eta \in \xi - \{0\}\rangle)$ be an arbitrary element of $\mathbb{P}$.

By induction on ξ, we can easily see:

> Assume ξ is even or limit and $\langle T_\eta \mid \eta \in \xi - \{0\}\rangle$ is an initial play in which II has played according to $\bar\tau$. Let S, T be as in the definition of $\bar\tau$. Then:
>
> i) For each $t \in S$, $\langle Suc_{T_\eta}(t) \mid \eta \in \xi - \{0\}\rangle$ is a play in $\Omega_\delta(\mathbb{P}_I)$ in which Player II plays according to τ.
>
> ii) $T \in \mathbb{P}$ and $T \leq T_\eta$ for every $\eta \in \xi - \{0\}$.

In particular, ii) implies that $\bar\tau$ is a winning strategy for Player II in $\Omega_\delta(\mathbb{P})$. □

Theorem 3.10. *Let κ be a regular uncountable cardinal and let I be a normal ideal on κ.*

(1) (Prikry [9]) *Assume that $\delta < \kappa$ and I is δ-saturated. Let Γ be (V, PR^*_I)-generic. Then for each $\alpha, \beta < \kappa$ and $f \in {}^\alpha\beta \cap V[\Gamma]$, there is an $F \in {}^\alpha\mathcal{P}(\beta) \cap V$ such that for every $\xi \in \alpha$, $|F(\xi)|^V < \delta$ and $f(\xi) \in F(\xi)$.* (*We say that $F < \delta$-covers f.*)

(2) (Shelah [10]) *Assume that $\omega < \delta \leq \kappa$ and I is δ-strategically closed. Then PR^+_I does not add any bounded subset of δ.*

Proof. (1). Assume not. Then there are α, $\beta \in \kappa$, $\langle t, T\rangle \in PR^*_I$ and a PR^*_I-name $\dot f$ such that $\langle t, T\rangle$ forces $\dot f \in {}^\alpha\beta$ and there is no $F \in V$ which $< \delta$-covers $\dot f$. Let $\mathbb{P}_\omega$ be the direct limit of $\langle \mathbb{P}_{I^n} \mid n \in \omega\rangle$ with respect to the natural complete embeddings and let G_ω be $(V, \mathbb{P}_\omega)$-generic. Let $\langle M_n, j_{m,n} \mid m \leq n \leq \omega\rangle$ be as in Theorem 3.3. Let $M = M_\omega$, $j := j_{0,\omega}$ and $\vec\kappa = \langle j_{0,n}(\kappa) \mid n \in \omega\rangle$.

We work in $V[G_\omega]$. By Theorem 3.3 and Lemma 3.2, $t^\frown\vec\kappa$ is a $PR^*_{j(I)}$-sequence over M. Let Γ_t be the $(M, PR^*_{j(I)})$-generic filter generated by $t^\frown\vec\kappa$ and let f be the interpretation of $j(\dot f)$ by Γ_t. Note that $j(\langle t, T\rangle) \in \Gamma_t$ because $j(t) = t$ and $\vec\kappa \restriction n \in j(T)$ for each $n \in \omega$. So, because j is elementary and does not move α, β and δ, $f \in {}^\alpha\beta$ and there is no $F \in M$ which δ-covers f. On the other hand, because $\mathbb{P}_\omega$ has the δ-c.c. and $f \in V[G_\omega]$, there is an $F \in {}^\alpha\mathcal{P}(\beta) \cap V$ which $< \delta$-covers f. Then $F = j(F) \in M$. This is a contradiction.

(2). We show (2) by almost the same argument. Assume the contrary. Then there is an $\alpha < \delta$, a PR^+_I-name $\dot x$ and $\langle t, T\rangle \in PR^+_I$ such that $\langle t, T\rangle$ forces $\dot x \subseteq \alpha$ and $\dot x \notin V$. Let $\mathbb{P}_\omega$ be the inverse limit of $\langle \mathbb{P}_{I^n} \mid n \in \omega\rangle$ with respect to the natural projections and let G_ω be a $(V, \mathbb{P}_\omega)$-generic filter such that $\langle T_{(n)} \mid n \in \omega\rangle \in G_\omega$. Let $\langle M_n, j_{m,n} \mid m \leq n \leq \omega\rangle$ be as in Theorem 3.5. Let $M = M_\omega$, $j := j_{0,\omega}$ and $\vec\kappa = \langle j_{0,n}(\kappa) \mid n \in \omega\rangle$.

Work in $V[G_\omega]$. Let Γ_t be a $(M, PR^+_{j(I)})$-generic filter generated by $t^\frown\vec\kappa$. Let x be the interpretation of $j(\dot x)$ by Γ_t. Then because $\langle T_{(n)} \mid n \in \omega\rangle \in G_\omega$, $\vec\kappa \restriction n \in j(T)$ for each $n \in \omega$ and so $j(\langle t, T\rangle) \in \Gamma_t$. Then, by the elementarity of j, $x \subseteq \alpha$ and $x \notin M$. On the other hand, because $\mathbb{P}_\omega$ is δ-strategically closed, $x \in V$. Because $x \subseteq \alpha < \kappa$, $x = j(x) \in M$. This is a contradiction. □

Next we discuss semiproperness of PR^* and PR^+. We begin with a review of semiproperness of posets.

A poset $\mathbb{P}$ is called *semiproper* if for every cardinal $\lambda > 2^{2^{|tcl(\mathbb{P})|}}$ there is a club $C \subseteq [\mathcal{H}_\lambda]^\omega$ such that for every $N \in C$ and $p \in \mathbb{P} \cap N$, there is a $p^* \leq p$ which forces "$N[\dot{G}] \cap \omega_1^V = N \cap \omega_1^V$". Here $\dot{G}$ is the canonical name of a $\mathbb{P}$-generic filter and, for each $(V, \mathbb{P})$-generic filter G, $N[G] := \{\dot{x}_G \mid \dot{x} \text{ is a } \mathbb{P}\text{-name} \wedge \dot{x} \in N\}$. We call the above p^* a semimaster condition for N. Recall that $\mathbb{P}$ is semiproper if and only if for "some" cardinal $\lambda > 2^{2^{|tcl(\mathbb{P})|}}$ there is a club $C \subseteq [\mathcal{H}_\lambda]^\omega$ such that for every $N \in C$ and $p \in \mathbb{P} \cap N$, there is a semimaster condition for N below p. (See Shelah [10].)

Now we give new conditions for PR_I^* and PR_I^+ being semiproper, which are related to the iteration of generic ultrapowers.

Theorem 3.11. *Let κ be a regular uncountable cardinal and I be a normal precipitous ideal on κ. Let $\sigma_{m,n} : \mathbb{P}_{I^m} \to \mathbb{P}_{I^n}$ be the natural complete embedding and $\pi_{n,m} : \mathbb{P}_{I^n} \to \mathbb{P}_{I^m}$ be the natural projection for $m \leq n \in \omega$.*

1. *If the direct limit of $\langle \mathbb{P}_{I^n}, \sigma_{m,n} \mid m \leq n \in \omega \rangle$ is semiproper then PR_I^* is semiproper.*
2. *If the inverse limit of $\langle \mathbb{P}_{I^n}, \pi_{n,m} \mid m \leq n \in \omega \rangle$ is semiproper then PR_I^+ is semiproper.*

Proof. (2) can be shown in the same way as (1). So we show only (1). Let $\mathbb{P}_\omega$ be the direct limit of $\langle \mathbb{P}_{I^n}, \sigma_{m,n} \mid m \leq n \in \omega \rangle$.

In $V^{\mathbb{P}_\omega}$, let $\langle M_n, j_{m,n} \mid m \leq n \leq \omega \rangle$ be as in Theorem 3.3 and let $M = M_\omega$, $j = j_{0,\omega}$. For each $t \in {}^{<\omega}\kappa$, let $\dot{\Gamma}_t$ be a $\mathbb{P}_\omega$-name for the $(M, PR_{j(I)}^*)$-generic filter generated by $t^\frown \langle j_{0,n}(\kappa) \mid n \in \omega \rangle$. Note that if $\dot{x} \in V$ is a PR_I^*-name then $j(\dot{x}) \in M$ is a $PR_{j(I)}^*$-name. So there is a $\mathbb{P}_\omega$-name $\dot{a} \in V$ such that

$$V^{\mathbb{P}_\omega} \models \text{"} \dot{a} = j(\dot{x})_{\dot{\Gamma}_t} := \text{the interpretation of } j(\dot{x}) \text{ by } \dot{\Gamma}_t \text{"}$$

In V, let λ be a cardinal such that

- $\kappa, I, \mathbb{P}_\omega, PR_I^* \in \mathcal{H}_\lambda$,
- If $\dot{x} \in \mathcal{H}_\lambda$ is a PR_I^*-name and $t \in {}^{<\omega}\kappa$ then there is a $\mathbb{P}_\omega$-name $\dot{a} \in \mathcal{H}_\lambda$ such that $V^{\mathbb{P}_\omega} \models$ "$\dot{a} = j(\dot{x})_{\dot{\Gamma}_t}$" .

Let $F : \mathcal{H}_\lambda \to \mathcal{H}_\lambda$ be a function witnessing the second condition above, i.e. for each PR_I^*-name $\dot{x} \in \mathcal{H}_\lambda$ and $t \in {}^{<\omega}\kappa$, $F(\dot{x}, t)$ is a $\mathbb{P}_\omega$-name such that $V^{\mathbb{P}_\omega} \models$ "$F(\dot{x}, t) = j(\dot{x})_{\dot{\Gamma}_t}$". Let $C \subseteq [\mathcal{H}_\lambda]^\omega$ be a club witnessing the properness of $\mathbb{P}_\omega$.

We show that if N is a countable elementary submodel of $\langle \mathcal{H}_\lambda, \in, \kappa, I, F, C \rangle$ and $\langle t, T \rangle \in N \cap PR_I^*$ then there is a semimaster condition for N below $\langle t, T \rangle$. Let N and $\langle t, T \rangle$ be as above. Because $N \in C$ and C witnesses the semiproperness of $\mathbb{P}_\omega$ there is a semimaster condition $p \in \mathbb{P}_\omega$ for N. Let G_ω be a $(V, \mathbb{P}_\omega)$-generic filter containing p. In $V[G_\omega]$, let j and M be as above. Note that ω_1 is absolute among V, M and $V[G_\omega]$.

We work in $V[G_\omega]$. Let Γ_t be the interpretation of $\dot{\Gamma}_t$ by G_ω. Because N is countable in V, $j(N) = j[N]$. Then, because $t \in N$ and N is closed under F,

$$\begin{aligned} j(N)[\Gamma_t] &= \{\dot{y}_{\Gamma_t} \mid \dot{y} \in j(N) \wedge \dot{y} \text{ is a } PR^*_{j(I)}\text{-name}\} \\ &= \{j(\dot{x})_{\Gamma_t} \mid \dot{x} \in N \wedge \dot{x} \text{ is a } PR^*_I\text{-name}\} \\ &= \{F(\dot{x}, t)_{G_\omega} \mid \dot{x} \in N \wedge \dot{x} \text{ is a } PR^*_I\text{-name}\} \\ &\subseteq N[G_\omega] \ . \end{aligned}$$

So, because $j \restriction \omega_1 = id$ and G_ω contains p,

$$j(N)[\Gamma_t] \cap \omega_1 \subseteq N[G_\omega] \cap \omega_1 = N \cap \omega_1 \subseteq j(N) \cap \omega_1$$

and thus $j(N)[\Gamma_t] \cap \omega_1 = j(N) \cap \omega_1$. This implies that there is a semimaster condition for $j(N)$ in Γ_t. Note that $j(\langle t, T \rangle) \in \Gamma_t$. Therefore $M_\omega \models$ "there is a semimaster condition for $j(N)$ below $j(\langle t, T \rangle)$". So, by the elementarity of j, $V \models$"there is a semimaster condition for N below $\langle t, T \rangle$". □

From Theorem 3.11 and Lemma 3.9, we can obtain the following corollary immediately. We believe that (1) of the following corollary is already known, too.

Corollary 3.12. *Let κ be a regular uncountable cardinal and I be a normal ideal on κ.*

(1) *If I is ω_1-saturated then PR^*_I is semiproper.*

(2) (Shelah [10]) *If I is $\omega + 1$-strategically closed then PR^+_I is semiproper.*

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