Katětov and Katětov-Blass orders on F_σ-ideals

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Abstract

We study the structures $(F_\sigma\,\mathsf{ideals}, \leq_{\mathrm{K}})$ and $(F_\sigma\,\mathsf{ideals}, \leq_{\mathrm{KB}})$, where $F_\sigma\,\mathsf{ideals}$ is the family of all F_σ-ideals over ω, and $\leq_{\mathrm{K}}$ and $\leq_{\mathrm{KB}}$ denote the Katětov and Katětov-Blass orders on ideals. We prove the following:

- $(F_\sigma\,\mathsf{ideals}, \leq_{\mathrm{K}})$ and $(F_\sigma\,\mathsf{ideals}, \leq_{\mathrm{KB}})$ are upward directed.
- The least cardinalities of cofinal subfamilies of $(F_\sigma\,\mathsf{ideals}, \leq_{\mathrm{K}})$ and $(F_\sigma\,\mathsf{ideals}, \leq_{\mathrm{KB}})$ are both equal to $\mathfrak{d}$. Moreover those of unbounded subfamilies are both equal to $\mathfrak{b}$.
- The family of all summable ideals is unbounded in $(F_\sigma\,\mathsf{ideals}, \leq_{\mathrm{K}})$ and $(F_\sigma\,\mathsf{ideals}, \leq_{\mathrm{KB}})$.

1 Introduction

In this paper an *ideal* means a proper ideal over ω including all finite subsets of ω. By identifying subsets of ω with their characteristic functions, an ideal can be naturally seen as a subset of the Cantor space 2^ω. An F_σ*-ideal* is an ideal which is F_σ as a subset of 2^ω. Let $F_\sigma\,\mathsf{ideals}$ denote the family of all F_σ-ideals. A typical example of an F_σ-ideal is a summable ideal: An ideal $\mathcal{I}$ is called a *summable ideal* if there exists a function $p : \omega \to \mathbb{Q}_{\geq 0}$ with $\sum_{k \in \omega} p(k) = \infty$ such that

$$\mathcal{I} \;=\; \left\{ A \subset \omega : \sum_{k \in A} p(k) < \infty \right\} ,$$

where $\mathbb{Q}_{\geq 0}$ denotes the set of all non-negative rational numbers. [1] The family of all summable ideals is denoted by $\mathsf{summable\ ideals}$.

In [5], Laflamme tries to destroy mad families with least damages and develops two methods for diagonalizing F_σ-filters. As one of results, he proves that under CH there exists a mad family $\mathcal{A}$ such that $\mathcal{A}$ is not included in any F_σ-ideal. Such a mad family is called a *Laflamme family.* It is not yet known whether the existence of a Laflamme family is provable in ZFC.

To construct a Laflamme family under weaker assumptions than CH, it may look important to investigate the structure $(F_\sigma\,\mathsf{ideals}, \subseteq)$, because it is enough

[1] It is usual to define summable ideals with $p : \omega \to \mathbb{R}_{\geq 0}$. Given $p : \omega \to \mathbb{R}_{\geq 0}$, choose $p^* : \omega \to \mathbb{Q}_{\geq 0}$ such that $|p(n) - p^*(n)| \leq \frac{1}{2^n}$ for $n \in \omega$. Then $\left\{A \subset \omega : \sum_{n \in A} p(n) < \infty\right\} = \left\{A \subset \omega : \sum_{n \in A} p^*(n) < \infty\right\}$.

to take care of a cofinal subfamily of $(F_\sigma\text{ideals}, \subseteq)$ instead of all F_σ-ideals. (See Section 5 for more details.) But this does not seem to be so useful because any cofinal subfamily has the same cardinality as the whole $F_\sigma\text{ideals}$:

Proposition 1.1. *The cardinality of any cofinal subfamily of* $(F_\sigma\text{ideals}, \subseteq)$ *is equal to* $\mathfrak{c}$.

Proof. Let $\mathcal{A}$ be an almost disjoint family with cardinality $\mathfrak{c}$. For each $A \in \mathcal{A}$ let $\mathcal{I}_A$ be the ideal on ω generated by $\omega \setminus A$ (and all finite subsets of ω). It is easy to see that $\mathcal{I}_A$ is an F_σ-ideal.

Note that for any distinct $A, B \in \mathcal{A}$ there is no (proper) ideal $\mathcal{I}$ with $\mathcal{I} \supseteq \mathcal{I}_A, \mathcal{I}_B$. (If $\mathcal{I}_A, \mathcal{I}_B \subseteq \mathcal{I}$, then $\mathcal{I} \ni (\omega \setminus A) \cup (\omega \setminus B) = \omega \setminus (A \cap B)$, and so $\omega \in \mathcal{I}$.) Hence the cardinality of any cofinal subfamily of $(F_\sigma\text{ideals}, \subseteq)$ is greater than or equal to $|\mathcal{A}| = \mathfrak{c}$. □

The Katětov order $\leq_{\mathrm{K}}$ and *the Katětov-Blass order* $\leq_{\mathrm{KB}}$ on ideals are refinements of the relation $\subseteq$, which are defined as follows:

- $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$ if there exists a function $\tau : \omega \to \omega$ such that $\tau^{-1}[A] \in \mathcal{J}$ for all $A \in \mathcal{I}$.
- $\mathcal{I} \leq_{\mathrm{KB}} \mathcal{J}$ if there exists a finite to one function $\tau : \omega \to \omega$ such that $\tau^{-1}[A] \in \mathcal{J}$ for all $A \in \mathcal{I}$.

Notice that for all ideals $\mathcal{I}$ and $\mathcal{J}$, $\mathcal{I} \subset \mathcal{J}$ implies $\mathcal{I} \leq_{\mathrm{KB}} \mathcal{J}$, and $\mathcal{I} \leq_{\mathrm{KB}} \mathcal{J}$ implies $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$. So a mad family $\mathcal{A}$ is a Laflamme family if there is no F_σ-ideal above $\mathcal{A}$ with respect to $\leq_{\mathrm{K}}$. To construct a Laflamme family under weaker assumptions than CH, it may be helpful to study the structures $(F_\sigma\text{ideals}, \leq_{\mathrm{K}})$ and $(F_\sigma\text{ideals}, \leq_{\mathrm{KB}})$.

In this paper we study basic properties of the structures $(F_\sigma\text{ideals}, \leq_{\mathrm{K}})$ and $(F_\sigma\text{ideals}, \leq_{\mathrm{KB}})$. First we prove the following:

Theorem 1. $(F_\sigma\text{ideals}, \leq_{\mathrm{K}})$ *and* $(F_\sigma\text{ideals}, \leq_{\mathrm{KB}})$ *are upward directed.*

Next we investigate the cofinal types of $(F_\sigma\text{ideals}, \leq_{\mathrm{K}})$ and $(F_\sigma\text{ideals}, \leq_{\mathrm{KB}})$. For this we use the notion of generalized Galois-Tukey connections introduced by Vojtáš [10]. First we recall them. In this paper we follow the formulation and terminology of Blass [2]:

Let $\mathbf{A} = (A_-, A_+, A)$ and $\mathbf{B} = (B_-, B_+, B)$ be triples such that A is a binary relation between A_- and A_+ and B is a binary relation between B_- and B_+. A *morphism* from $\mathbf{A}$ to $\mathbf{B}$ is a pair $\rho = (\rho_-, \rho_+)$ of functions such that

- $\rho_- : B_- \to A_-$,
- $\rho_+ : A_+ \to B_+$,
- for all $b \in B_-$ and $a \in A_+$, $\rho_-(b) A a$ implies $b B \rho_+(a)$.

$Y \subseteq A_+$ is said to be A-*cofinal* if for any $x \in A_-$ there is $y \in Y$ with xAy, and $X \subseteq A_-$ is said to be A-*unbounded* if for any $y \in A_+$ there is $x \in X$ with $\neg xAy$. Note that if $\rho = (\rho_-, \rho_+)$ is a morphism from $\mathbf{A}$ to $\mathbf{B}$, then $\rho_-[X]$ is A-unbounded for any B-unbounded $X \subseteq B_-$, and $\rho_+[Y]$ is B-cofinal for any A-cofinal $Y \subseteq A_+$.

Let $\mathbf{B} \preceq \mathbf{A}$ denote that there is a morphism from $\mathbf{A}$ to $\mathbf{B}$. Notice that $\preceq$ is transitive. We write $\mathbf{A} \equiv \mathbf{B}$ if $\mathbf{A} \preceq \mathbf{B}$, and $\mathbf{B} \preceq \mathbf{A}$.

If $A_- = A_+$, then $\mathbf{A} = (A_-, A_+, A)$ is simply denoted as (A_-, A). So $(\omega^\omega, \leq^*)$, $(F_\sigma\textsf{ideals}, \leq_{\mathrm{K}})$ and $(F_\sigma\textsf{ideals}, \leq_{\mathrm{KB}})$ respectively denote $(\omega^\omega, \omega^\omega, \leq^*)$, $(F_\sigma\textsf{ideals}, F_\sigma\textsf{ideals}, \leq_{\mathrm{K}})$ and $(F_\sigma\textsf{ideals}, F_\sigma\textsf{ideals}, \leq_{\mathrm{KB}})$. Define the dual $\mathbf{A}^\perp$ of $\mathbf{A} = (A_-, A_+, A)$ by $\mathbf{A}^\perp = (A_+, A_-, A^\perp)$, where

$$A^\perp = \{(y, x) \in A_+ \times A_- : \neg xAy\} .$$

If $\rho = (\rho_-, \rho_+)$ witnesses $\mathbf{A} \preceq \mathbf{B}$, then $\rho^\perp = (\rho_+, \rho_-)$ witnesses $\mathbf{B}^\perp \preceq \mathbf{A}^\perp$.

The norm $\|\mathbf{A}\|$ is the least cardinality of an A-cofinal subset of A_+. Notice that $\|\mathbf{A}^\perp\|$ is the least cardinality of an A-unbounded subset of A_-. Notice also that if $\mathbf{B} \preceq \mathbf{A}$, then $\|\mathbf{B}\| \leq \|\mathbf{A}\|$, and $\|\mathbf{B}^\perp\| \geq \|\mathbf{A}^\perp\|$.

Now we present our results. We prove the following:

Theorem 2.

$$\begin{aligned} &(F_\sigma\textsf{ideals}, \leq_{\mathrm{K}}) \equiv (F_\sigma\textsf{ideals}, \leq_{\mathrm{KB}}) \\ &\equiv (\textsf{summable ideals}, F_\sigma\textsf{ideals}, \leq_{\mathrm{K}}) \equiv (\textsf{summable ideals}, F_\sigma\textsf{ideals}, \leq_{\mathrm{KB}}) \\ &\equiv (\omega^\omega, \leq^*). \end{aligned}$$

From Theorem 2 we obtain that

$$\begin{aligned} &\|(F_\sigma\textsf{ideals}, \leq_{\mathrm{K}})\| = \|(F_\sigma\textsf{ideals}, \leq_{\mathrm{KB}})\| = \|(\omega^\omega, \leq^*)\| , \\ &\|(F_\sigma\textsf{ideals}, \leq_{\mathrm{K}})^\perp\| = \|(F_\sigma\textsf{ideals}, \leq_{\mathrm{KB}})^\perp\| = \|(\omega^\omega, \leq^*)^\perp\| , \end{aligned}$$

which mean the following:

Corollary 3.

(1) *The least cardinalities of cofinal subfamilies of* $(F_\sigma\textsf{ideals}, \leq_{\mathrm{K}})$ *and* $(F_\sigma\textsf{ideals}, \leq_{\mathrm{KB}})$ *are both equal to* $\mathfrak{d}$.

(2) *The least cardinalities of unbounded subfamilies of* $(F_\sigma\textsf{ideals}, \leq_{\mathrm{K}})$ *and* $(F_\sigma\textsf{ideals}, \leq_{\mathrm{KB}})$ *are both equal to* $\mathfrak{b}$.

Moreover from the facts that

$$\begin{aligned} &(F_\sigma\textsf{ideals}, \leq_{\mathrm{K}}) \preceq (\textsf{summable ideals}, F_\sigma\textsf{ideals}, \leq_{\mathrm{K}}) , \\ &(F_\sigma\textsf{ideals}, \leq_{\mathrm{KB}}) \preceq (\textsf{summable ideals}, F_\sigma\textsf{ideals}, \leq_{\mathrm{KB}}) , \end{aligned}$$

we obtain the following:

Corollary 4. *The family of all summable ideals is unbounded in* $(F_\sigma\textsf{ideals}, \leq_{\mathrm{K}})$ *and* $(F_\sigma\textsf{ideals}, \leq_{\mathrm{KB}})$.

This paper is organized as follows: We prove Theorem 1 in Section 3, and Theorem 2 is proved in Section 4. In the proof of these theorems we will use Mazur's characterization of F_σ-ideals using submeasures. In Section 2 we review this characterization and present basic facts on submeasures. In Section 5 we will present several questions which are relevant to our results.

2 Submeasures and F_σ-ideals

We will use Mazur's characterization of F_σ-ideals using submeasures. Here we review this characterization and present basic facts on submeasures.

A submeasure on a set X is a function $\varphi : [X]^{<\omega} \to \mathbb{Q}_{\geq 0}$ with the following properties:

(i) $\varphi(A) \leq \varphi(B)$ if $A \subseteq B$. (Monotonicity)

(ii) $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$. (Subadditivity)

(iii) $\varphi(\emptyset) = 0$.

If φ is a submeasure on X, then X is denoted by $\delta(\varphi)$. Note that there is a unique submeasure on $\emptyset$, which assigns $\emptyset$ to 0. Let φ_{trivial} be this submeasure.

If p is a function from X to $\mathbb{Q}_{\geq 0}$, then the function $\varphi : [X]^{<\omega} \to \mathbb{Q}_{\geq 0}$ defined by $\varphi(A) = \sum_{k \in A} p(k)$ is a submeasure. In fact, this φ is a measure, that is, φ satisfies (i), (iii) and (iv) below:

(iv) $\varphi(A \cup B) = \varphi(A) + \varphi(B)$ if A and B are disjoint. (Additivity)

We call this φ *the measure induced by p.*

Let φ be a submeasure on ω. For $A \subseteq \omega$ let

$$\hat{\varphi}(A) = \lim_{n \to \omega} \varphi(A \cap n) \in \mathbb{R}_{\geq 0} \cup \{\infty\} .$$

We say that φ is *unbounded* if $\hat{\varphi}(\omega) = \infty$. If φ is unbounded, then

$$\mathsf{Fin}(\varphi) = \{A \subseteq \omega : \hat{\varphi}(A) < \infty\} .$$

is an ideal. [2] Mazur's characterization of F_σ-ideals is as follows:

Theorem 2.1 (Mazur [6]). *The following are equivalent for $\mathcal{I}$:*

(i) $\mathcal{I}$ is an F_σ-ideal.

(ii) $\mathcal{I} = \mathsf{Fin}(\varphi)$ for some unbounded submeasure φ on ω.

[2]Recall that in this paper an "ideal" means a proper ideal over ω. Unboundedness of φ assures properness of $\mathsf{Fin}(\varphi)$.

In this paper we will use the above theorem without any notice. Next we present several notions and lemmata on submeasures, which will be used in this paper.

For submeasures φ_0 and φ_1 on a set X let

$$\varphi_0 \le \varphi_1 \overset{\text{def}}{\Leftrightarrow} \varphi_0(A) \le \varphi_1(A) \text{ for all } A \in [X]^{<\omega}.$$

Moreover for submeasures φ_0 on X_0 and φ_1 on X_1 let $\varphi_0 \curlywedge \varphi_1$ be the function on $[X_0 \cup X_1]^{<\omega}$ defined as

$$\varphi_0 \curlywedge \varphi_1(A) = \min\{\varphi_0(A_0) + \varphi_1(A_1) : A_0 \cup A_1 = A,\ A_0 \subseteq X_0,\ A_1 \subseteq X_1\}$$

for each $A \in [X_0 \cup X_1]^{<\omega}$.

Lemma 2.2. *Let φ_0 and φ_1 be submeasures on sets X_0 and X_1, respectively, and let $\varphi = \varphi_0 \curlywedge \varphi_1$.*

(1) φ is a submeasure on $X_0 \cup X_1$.

(2) $\varphi \restriction [X_0]^{<\omega} \le \varphi_0$, and $\varphi \restriction [X_1] \le \varphi_1$. Hence if $X_0 = X_1$, then $\varphi \le \varphi_0, \varphi_1$.

(3) If $X \subseteq X_0 \cap X_1$, and $\varphi_0 \restriction [X]^{<\omega} \le \varphi_1 \restriction [X]^{<\omega}$, then $\varphi \restriction [X]^{<\omega} = \varphi_0 \restriction [X]^{<\omega}$.

(4) $\varphi \restriction [X_0 \setminus X_1]^{<\omega} = \varphi_0 \restriction [X_0 \setminus X_1]^{<\omega}$, and $\varphi \restriction [X_1 \setminus X_0]^{<\omega} = \varphi_1 \restriction [X_1 \setminus X_0]^{<\omega}$.

Proof. (1) Clearly $\varphi(\emptyset) = 0$. We check the monotonicity and the subadditivity.

First we check the monotonicity. Suppose that $A \subseteq B \in [X_0 \cup X_1]^{<\omega}$. Take $B_0 \subseteq X_0$ and $B_1 \subseteq X_1$ such that $B_0 \cup B_1 = B$ and $\varphi(B) = \varphi_0(B_0) + \varphi_1(B_1)$. Let $A_i = A \cap B_i$ for each $i = 0, 1$. Note that $A = A_0 \cup A_1$. Then

$$\varphi(A) \ \le\ \varphi_0(A_0) + \varphi_1(A_1) \ \le\ \varphi_0(B_0) + \varphi_1(B_1) \ =\ \varphi(B)\,,$$

where the second inequality follows from the monotonicity of φ_0 and φ_1.

Next we check the subadditivity. Suppose that $A, B \in [X_0 \cup X_1]^{<\omega}$. Take $A_0, B_0 \subseteq X_0$ and $A_1, B_1 \subseteq X_1$ such that $A_0 \cup A_1 = A$, $\varphi(A) = \varphi_0(A_0) + \varphi_1(A_1)$, $B_0 \cup B_1 = B$ and $\varphi(B) = \varphi_0(B_0) + \varphi_1(B_1)$. Note that $(A_0 \cup B_0) \cup (A_1 \cup B_1) = A \cup B$. Then

$$\begin{aligned}\varphi(A \cup B) \ &\le\ \varphi_0(A_0 \cup B_0) + \varphi_1(A_1 \cup B_1)\\ &\le\ \varphi_0(A_0) + \varphi_0(B_0) + \varphi_1(A_1) + \varphi_1(B_1) \ =\ \varphi(A) + \varphi(B)\,,\end{aligned}$$

where the second inequality follows from the subadditivity of φ_0 and φ_1.

(2) By symmetry it suffices to prove the former. For any $A \in [X_0]^{<\omega}$ we have that $\varphi(A) \le \varphi_0(A) + \varphi_1(\emptyset) = \varphi_0(A)$.

(3) Suppose that $A \in [X]^{<\omega}$. Then $\varphi(A) \le \varphi_0(A)$ by (2). Moreover for each $A_0, A_1 \subseteq A$ with $A_0 \cup A_1 = A$ it holds that

$$\varphi_0(A_0) + \varphi_1(A_1) \ \ge\ \varphi_0(A_0) + \varphi_0(A_1) \ \ge\ \varphi_0(A)\,.$$

So $\varphi(A) \geq \varphi_0(A)$. Therefore $\varphi(A) = \varphi_0(A)$.

(4) It suffices to prove the former by symmetry. Suppose that $A \in [X_0 \setminus X_1]^{<\omega}$. Note that $A_0 = A$ and $A_1 = \emptyset$ if $A_0 \cup A_1 = A$, $A_0 \subseteq X_0$ and $A_1 \subseteq X_1$. Then $\varphi(A) = \varphi_0(A) + \varphi_1(\emptyset) = \varphi_0(A)$. □

Let φ be a submeasure on a set X. For $q \in \mathbb{Q}_{\geq 0}$ let $\varphi + q$ be the function on $[X]^{<\omega}$ such that

$$\varphi + q\,(A) \;=\; \begin{cases} \varphi(A) + q & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset. \end{cases}$$

It is easy to check that $\varphi + q$ is a submeasure on X. For a set Y and a function $\tau : Y \to X$ let $\varphi \circ \tau$ denote the pullback of φ by τ, that is, $\varphi \circ \tau$ is a function on $[Y]^{<\omega}$ such that

$$\varphi \circ \tau\,(A) \;=\; \varphi(\tau[A])\ .$$

Again it is easy to see that $\varphi \circ \tau$ is a submeasure on Y. The following lemma is clear from the definition of the Katětov and the Katětov-Blass orders:

Lemma 2.3. *Suppose that φ and ψ are unbounded submeasures on ω.*

(1) If there exist a function $\tau : \omega \to \omega$ and $q \in \mathbb{Q}_{\geq 0}$ such that $\psi \leq (\varphi + q) \circ \tau$, then $\mathsf{Fin}(\varphi) \leq_{\mathrm{K}} \mathsf{Fin}(\psi)$.

(2) If there exist a finite to one function $\tau : \omega \to \omega$ and $q \in \mathbb{Q}_{\geq 0}$ such that $\psi \leq (\varphi + q) \circ \tau$, then $\mathsf{Fin}(\varphi) \leq_{\mathrm{KB}} \mathsf{Fin}(\psi)$.

3 Directedness

In this section first we prove Theorem 1. After that we also observe that the family of all analytic P-ideals is upward directed with respect to $\leq_{\mathrm{K}}$ and $\leq_{\mathrm{KB}}$.

Theorem 1. *$(F_\sigma\,\mathsf{ideals}, \leq_{\mathrm{K}})$ and $(F_\sigma\,\mathsf{ideals}, \leq_{\mathrm{KB}})$ are upward directed.*

Proof. It suffices to prove that $(F_\sigma\,\mathsf{ideals}, \leq_{\mathrm{KB}})$ is upward directed because $\leq_{\mathrm{KB}}$ is coarser than $\leq_{\mathrm{K}}$. Suppose that $\mathcal{I}_0$ and $\mathcal{I}_1$ are F_σ-ideals. We must find an F_σ-ideal $\mathcal{J}$ such that $\mathcal{J} \geq_{\mathrm{KB}} \mathcal{I}_0, \mathcal{I}_1$.

For each $i = 0, 1$ let φ_i be an unbounded submeasure on ω such that $\mathcal{I}_i = \mathsf{Fin}(\varphi_i)$. By induction on $n \in \omega$ take $s_n \in \omega$ so that $s_0 = 0$ and so that $\varphi_i(s_{n+1} \setminus s_n) > n$ for both $i = 0, 1$. Let

$$A_n = (s_{n+1} \setminus s_n) \times (s_{n+1} \setminus s_n) \;=\; \{(k, l) : s_n \leq k, l < s_{n+1}\}$$

for each $n \in \omega$, and let $X = \bigcup_{n\in\omega} A_n$. We will construct a submeasure ψ on X and finite to one functions $\tau_i : X \to \omega$ for $i = 0, 1$ such that $\psi(A_n) > n$ for each $n \in \omega$ and such that $\psi \leq \varphi_i \circ \tau_i$ for both $i = 0, 1$. If such ψ and τ_i's are constructed, then ψ can be identified with an unbounded submeasure on ω

through a bijection between X and ω, and $\mathcal{J} := \mathsf{Fin}(\psi) \geq_{\mathrm{KB}} \mathcal{I}_0, \mathcal{I}_1$ by Lemma 2.3.

For each $i = 0, 1$ let $\tau_i : X \to \omega$ be the i-th projection, that is, $\tau_0(k, l) = k$ and $\tau_1(k, l) = l$ for $(k, l) \in X$. Note that both τ_0 and τ_1 are finite to one. Let

$$\psi = (\varphi_0 \circ \tau_0) \curlywedge (\varphi_1 \circ \tau_1) .$$

By Lemma 2.2, ψ is a submeasure on X, and $\psi \leq \varphi_i \circ \tau_i$ for $i = 0, 1$. It remains to show that $\psi(A_n) > n$ for each $n \in \omega$. Fix $n \in \omega$, and suppose that $B_0 \cup B_1 = A_n$. It suffices to show that $\varphi_0 \circ \tau_0(B_0) + \varphi_1 \circ \tau_1(B_1) > n$. First note that either $\tau_0[B_0] = s_{n+1} \setminus s_n$, or $\tau_1[B_1] = s_{n+1} \setminus s_n$. This is because if $k_i \in (s_{n+1} \setminus s_n) \setminus \tau_i[B_i]$ for $i = 0, 1$, then $(k_0, k_1) \in A_n \setminus (B_0 \cup B_1)$, which contradicts that $A_n = B_0 \cup B_1$. Recall also that $\varphi_i(s_{n+1} \setminus s_n) > n$ for both $i = 0, 1$. So either $\varphi_0 \circ \tau_0(B_0) > n$, or $\varphi_1 \circ \tau_1(B_1) > n$. Hence $\varphi_0 \circ \tau_0(B_0) + \varphi_1 \circ \tau_1(B_1) > n$. □

Next we observe that the family of all analytic P-ideals, denoted as $\mathsf{Analytic\ Pideals}$, is also upward directed with respect to $\leq_{\mathrm{K}}$ and $\leq_{\mathrm{KB}}$.

First we briefly review analytic P-ideals: An ideal $\mathcal{I}$ is called a P-ideal if for every countable subset $\mathcal{C}$ of $\mathcal{I}$, there exists $A \in \mathcal{I}$ such that $C \subseteq^* A$ for every $C \in \mathcal{C}$. It is known, due to Solecki [9], that Analytic P-ideals have a similar characterization to F_σ-ideals. To present this characterization, we prepare notation. A submeasure φ on ω is said to be *bounded* if $\hat{\varphi}(\omega) < \infty$. For a submeasure φ on ω let

$$\mathsf{Exh}(\varphi) = \left\{ X \subseteq \omega : \lim_{n \to \omega} \hat{\varphi}(X \setminus n) = 0 \right\} .$$

Note that $\mathsf{Exh}(\varphi)$ is an ideal if $\lim_{n \to \omega} \hat{\varphi}(\omega \setminus n) > 0$. $\mathsf{Exh}(\varphi)$ is called *the exhaustive ideal* of φ. The characterization of analytic P-ideals is as follows:

Theorem 3.1 (Solecki [9])**.** *The following are equivalent for $\mathcal{I}$:*

(i) $\mathcal{I}$ is an analytic P-ideal.

(ii) $\mathcal{I} = \mathsf{Exh}(\varphi)$ for some bounded submeasure φ on ω with $\lim_{n \to \omega} \hat{\varphi}(\omega \setminus n) > 0$.

We prove the following:

Proposition 3.2. $(\mathsf{Analytic\ Pideals}, \leq_{\mathrm{K}})$ *and* $(\mathsf{Analytic\ Pideals}, \leq_{\mathrm{KB}})$ *are upward directed.*

Proof. The proof is almost the same as that of Theorem 1. Suppose that $\mathcal{I}_0$ and $\mathcal{I}_1$ are analytic P-ideals. It suffices to find an analytic P-ideal $\mathcal{J}$ with $\mathcal{I}_0, \mathcal{I}_1 \leq_{\mathrm{KB}} \mathcal{J}$.

For each $i = 0, 1$, let φ_i be a bounded submeasure on ω such that $\mathcal{I}_i = \mathsf{Exh}(\varphi_i)$. Let $c_i = \lim_{n \to \omega} \hat{\varphi}(\omega \setminus n) > 0$. By induction on $n \in \omega$, take $s_n \in \omega$ so that $s_0 = 0$ and $\varphi_i(s_{n+1} \setminus s_n) \geq c_i - \frac{1}{2^n}$ for both $i = 0, 1$. Let $A_n = (s_{n+1} \setminus s_n) \times (s_{n+1} \setminus s_n)$ for each $n \in \omega$, and let $X = \bigcup_{n \in \omega} A_n$. It suffices to find a submeasure ψ on X and finite to one functions $\tau_i : X \to \omega$ for $i = 0, 1$ such that $\liminf_{n\to\omega}\psi(A_n) > 0$ and $\psi \le \varphi_i \circ \tau_i$: Then ψ can be identified with a bounded submeasure on ω such that $\lim_{n\to\omega}\hat{\psi}(\omega\setminus n) > 0$. Here the boundedness follows from that of φ_0, and the fact that $\psi \le \varphi \circ \tau_0$. Then $\mathcal{J} = \mathsf{Exh}(\psi)$ is an analytic P-ideal, and $\mathcal{J} \ge_{\mathrm{KB}} \mathcal{I}_0, \mathcal{I}_1$.

For each $i = 0, 1$ let $\tau_i : X \to \omega$ be the i-th projection, and let

$$\psi = (\varphi_0 \circ \tau_0) \curlywedge (\varphi_1 \circ \tau_1) .$$

Then τ_i is finite to one, ψ is a submeasure on X, and $\psi \le \varphi_i \circ \tau_i$. Moreover for each $n \in \omega$, by the same argument as in the proof of Theorem 1,

$$\psi(A_n) \ge \min\{\varphi_0(s_{n+1} \setminus s_n), \varphi_1(s_{n+1} \setminus s_n)\} = \min\{c_0, c_1\} - \frac{1}{2^n} .$$

Thus $\liminf_{n\to\infty}\psi(A_n) \ge \min\{c_0, c_1\} > 0$. □

4 Cofinal types

In this section we prove Theorem 2:

Theorem 2.

$$(F_\sigma\mathsf{ideals}, \le_{\mathrm{K}}) \equiv (F_\sigma\mathsf{ideals}, \le_{\mathrm{KB}})$$
$$\equiv (\mathsf{summable\ ideals}, F_\sigma\mathsf{ideals}, \le_{\mathrm{K}}) \equiv (\mathsf{summable\ ideals}, F_\sigma\mathsf{ideals}, \le_{\mathrm{KB}})$$
$$\equiv (\omega^\omega, \le^*).$$

Note that the pair of the identity maps on $F_\sigma\mathsf{ideals}$ is a morphism from $(F_\sigma\mathsf{ideals}, \le_{\mathrm{KB}})$ to $(F_\sigma\mathsf{ideals}, \le_{\mathrm{K}})$ because $\le_{\mathrm{K}}$ refines $\le_{\mathrm{KB}}$. Similarly, the pair of the identity maps is a morphism from $(\mathsf{summable\ ideals}, F_\sigma\mathsf{ideals}, \le_{\mathrm{KB}})$ to $(\mathsf{summable\ ideals}, F_\sigma\mathsf{ideals}, \le_{\mathrm{K}})$. Note also that the pair of the identity maps is a morphism from $(F_\sigma\mathsf{ideals}, \le_{\mathrm{K}})$ to $(\mathsf{summable\ ideals}, F_\sigma\mathsf{ideals}, \le_{\mathrm{K}})$ and from $(F_\sigma\mathsf{ideals}, \le_{\mathrm{KB}})$ to $(\mathsf{summable\ ideals}, F_\sigma\mathsf{ideals}, \le_{\mathrm{KB}})$. So we have the following:

$$\begin{array}{ccc}
(\mathsf{summable\ ideals}, F_\sigma\mathsf{ideals}, \le_{\mathrm{K}}) & \preceq & (\mathsf{summable\ ideals}, F_\sigma\mathsf{ideals}, \le_{\mathrm{KB}}) \\
\rotatebox{90}{$\preceq$} & & \rotatebox{90}{$\preceq$} \\
(F_\sigma\mathsf{ideals}, \le_{\mathrm{K}}) & \preceq & (F_\sigma\mathsf{ideals}, \le_{\mathrm{KB}})
\end{array}$$

Hence it suffices to prove the following:

- $(F_\sigma\mathsf{ideals}, \le_{\mathrm{KB}}) \preceq (\omega^\omega, \le^*)$.
- $(\omega^\omega, \le^*) \preceq (\mathsf{summable\ ideals}, F_\sigma\mathsf{ideals}, \le_{\mathrm{K}})$.

These are proved in Proposition 4.1 and 4.6. First we prove the former:

Proposition 4.1. $(F_\sigma\mathsf{ideals}, \le_{\mathrm{KB}}) \preceq (\omega^\omega, \le^*)$.

We will define an upward directed $(H, \leq^\circ)$ and prove that

$$(F_\sigma \text{ideals}, \leq_{\mathrm{KB}}) \preceq (H, \leq^\circ) \preceq (\omega^\omega, \leq^*) .$$

First we define $(H, \leq^\circ)$. Let Φ be the set of all submeasures φ with $\delta(\varphi) \in \omega$. Note that Φ is countable. For $\varphi \in \Phi$ and $n \in \omega$ let $\Phi(\varphi, n)$ be the set of all $\bar{\varphi} \in \Phi$ such that

(i) $\bar{\varphi}$ extends φ, i.e. $\bar{\varphi} \restriction \mathcal{P}(\delta(\varphi)) = \varphi$,

(ii) $\bar{\varphi}(\delta(\bar{\varphi}) \setminus \delta(\varphi)) > n$.

Then let

$$H = \{h \in \Phi^{\Phi \times \omega} : \forall (\varphi, n) \in \Phi \times \omega,\ h(\varphi, n) \in \Phi(\varphi, n)\} .$$

To define the order $\leq^\circ$, first define an order $\trianglelefteq_\varphi$ on $\Phi(\varphi, n)$ as follows: $\bar{\varphi} \trianglelefteq_\varphi \bar{\varphi}'$ if there exists a function $\bar{\tau} : \delta(\bar{\varphi}') \to \delta(\bar{\varphi})$ such that

(i) $\bar{\tau} \restriction \delta(\varphi) = \mathrm{id}$, and $\bar{\tau}[\delta(\bar{\varphi}') \setminus \delta(\varphi)] \subseteq \delta(\bar{\varphi}) \setminus \delta(\varphi)$,

(ii) $\bar{\varphi}' \leq \bar{\varphi} \circ \bar{\tau}$.

It is easy to see that $\trianglelefteq_\varphi$ is transitive. Then for each $h, h' \in H$ let $h \leq^\circ h'$ if $h(\varphi, n) \trianglelefteq_\varphi h'(\varphi, n)$ for all but finitely many $(\varphi, n) \in \Phi \times \omega$. Note that $\leq^\circ$ is also transitive.

First we show that $(H, \leq^\circ)$ is upward directed. This immediately follows from the lemma below:

Lemma 4.2. *$(\Phi(\varphi, n), \trianglelefteq_\varphi)$ is upward directed for all $\varphi \in \Phi$ and $n \in \omega$.*

Proof. The proof is similar as that of Theorem 1. Suppose that $\bar{\varphi}_0, \bar{\varphi}_1 \in \Phi(\varphi, n)$. We must find their upper bound in $(\Phi(\varphi, n), \trianglelefteq_\varphi)$. Let $s = \delta(\varphi)$ and $\bar{s}_i = \delta(\bar{\varphi}_i)$ for $i = 0, 1$. Moreover let $A = (\bar{s}_0 \setminus s) \times (\bar{s}_1 \setminus s)$. It suffices to find a submeasure $\bar{\varphi}$ on $s \cup A$ and functions $\bar{\tau}_i : s \cup A \to \bar{s}_i$ for $i = 0, 1$ such that

(i) $\bar{\varphi}$ extends φ, and $\bar{\varphi}(A) > n$,

(ii) $\bar{\tau}_i \restriction s = \mathrm{id}$, and $\bar{\tau}_i[A] \subseteq \bar{s}_i \setminus s$,

(iii) $\bar{\varphi} \leq \bar{\varphi}_i \circ \bar{\tau}_i$.

Let $\bar{\tau}_i \restriction s = \mathrm{id}$, and let $\bar{\tau}_i \restriction A$ be the i-th projection. Clearly $\bar{\tau}_i$ satisfies (ii) above. Let

$$\bar{\varphi} = (\bar{\varphi}_0 \circ \bar{\tau}_0) \curlywedge (\bar{\varphi}_1 \circ \bar{\tau}_1) .$$

Then $\bar{\varphi}$ is a submeasure on $s \cup A$ satisfying (iii) by Lemma 2.2 (1) and (2). Here note that $(\bar{\varphi}_i \circ \bar{\tau}_i) \restriction \mathcal{P}(s) = \varphi$ for $i = 0, 1$. So $\bar{\varphi}$ extends φ by Lemma 2.2 (3). It remains to show that $\bar{\varphi}(A) > n$, but this can be proved by the same argument as in the proof of Theorem 1. $\square$

Using Lemma 4.2, we can easily prove the following:

Lemma 4.3. $(H, \leq^\circ) \preceq (\omega^\omega, \leq^*)$.

Proof. We will construct a morphism $\rho = (\rho_-, \rho_+)$ from $(\omega^\omega, \leq^*)$ to $(H, \leq^\circ)$. First take a one to one enumeration $((\varphi_i, n_i) : i \in \omega)$ of $\Phi \times \omega$. Moreover for each $i \in \omega$ take an enumeration $(\bar{\varphi}^i_j : j \in \omega)$ of $\Phi(\varphi_i, n_i)$. For each $h \in H$ let $\rho_-(h) \in \omega^\omega$ be such that $h(\varphi_i, n_i) = \bar{\varphi}^i_{\rho_-(h)(i)}$. For each $f \in \omega^\omega$ let $\rho_+(f) \in H$ be such that $\rho_+(f)(\varphi_i, n_i) \trianglerighteq_{\varphi_i} \bar{\varphi}^i_j$ for all $j \leq f(i)$. Note that there exists such $\rho_+(f)$ by Lemma 4.2.

To show that ρ is a morphism from $(\omega^\omega, \leq^*)$ to $(H, \leq^\circ)$, suppose that $h \in H$, $f \in \omega^\omega$ and $\rho_-(h) \leq^* f$. Note that for each $i \in \omega$ if $\rho_-(h)(i) \leq f(i)$, then $h(\varphi_i, n_i) \trianglelefteq_{\varphi_i} \rho_+(f)(\varphi_i, n_i)$. So $h \leq^\circ \rho_+(f)$ because $\rho_-(h) \leq^* f$. □

Next we will prove the following:

Lemma 4.4. $(F_\sigma \textsf{ideals}, \leq_{\mathrm{KB}}) \preceq (H, \leq^\circ)$.

For this we need the following lemma:

Lemma 4.5. *Suppose that $\varphi, \psi \in \Phi$. Let $n \in \omega$ and $\bar{\varphi} \in \Phi(\varphi, n)$. Then there is $\bar{\psi}^* \in \Phi(\psi, n)$ such that the following holds for all $\bar{\psi} \in \Phi(\psi, n)$ with $\bar{\psi}^* \trianglelefteq_\psi \bar{\psi}$: For any function $\tau : \delta(\psi) \to \delta(\varphi)$ and any $q \in \mathbb{Q}_{\geq 0}$ with $\psi \leq (\varphi + q) \circ \tau$ there exists a function $\bar{\tau} : \delta(\bar{\psi}) \to \delta(\bar{\varphi})$ such that*

(i) $\bar{\tau}$ extends τ, and $\bar{\tau}[\delta(\bar{\psi}) \setminus \delta(\psi)] \subseteq \delta(\bar{\varphi}) \setminus \delta(\varphi)$,

(ii) $\bar{\psi} \leq (\bar{\varphi} + q) \circ \bar{\tau}$.

Proof. Let s, $\bar{s}$ and t be $\delta(\varphi)$, $\delta(\bar{\varphi})$ and $\delta(\psi)$, respectively. Moreover let $\bar{t} = t + (\bar{s} - s)$. For each $\tau : t \to s$, let $\bar{\sigma}_\tau$, q_τ and $\bar{\psi}_\tau$ be as follows:

- $\bar{\sigma}_\tau : \bar{t} \to \bar{s}$ is the extension of τ such that $\bar{\sigma}_\tau(t + k) = s + k$.
- q_τ is the least non-negative rational number such that $\psi \leq (\varphi + q_\tau) \circ \tau$. (We can take such q_τ because ψ, φ and τ are all finite.)
- $\bar{\psi}_\tau = \psi \curlywedge ((\bar{\varphi} + q_\tau) \circ \bar{\sigma}_\tau)$.

Then $\bar{\psi}_\tau$ is a submeasure on $\bar{t}$ by Lemma 2.2 (1). Note that $(\bar{\varphi} + q_\tau) \circ \bar{\sigma}_\tau$ extends $(\varphi + q_\tau) \circ \tau$. So $\bar{\psi}_\tau$ extends ψ by Lemma 2.2 (3) and the fact that $\psi \leq (\varphi + q_\tau) \circ \tau$. Moreover, by Lemma 2.2 (4),

$$\bar{\psi}_\tau(\bar{t} \setminus t) \;=\; ((\bar{\varphi} + q_\tau) \circ \bar{\sigma}_\tau)(\bar{t} \setminus t) \;=\; \bar{\varphi}(\bar{s} \setminus s) + q_\tau \;>\; n\,.$$

Thus $\bar{\psi}_\tau \in \Phi(\psi, n)$ for each $\tau : t \to s$.

By Lemma 4.2 there is $\bar{\psi}^* \in \Phi(\psi, n)$ such that $\bar{\psi}^* \trianglerighteq_\psi \bar{\psi}_\tau$ for all $\tau : t \to s$. We claim that this $\bar{\psi}^*$ is as desired. Suppose that $\bar{\psi} \trianglerighteq_\psi \bar{\psi}^*$, and take an arbitrary function $\tau : \delta(\psi) \to \delta(\varphi)$ and an arbitrary $q \in \mathbb{Q}_{\geq 0}$ with $\psi \leq (\varphi + q) \circ \tau$. Then $\bar{\psi} \trianglerighteq_\psi \bar{\psi}_\tau$. Let $\bar{\sigma}' : \delta(\bar{\psi}) \to \delta(\bar{\psi}_\tau)$ be a witness of this relationship, and let $\bar{\tau}$ be the composition $\bar{\sigma}_\tau \circ \bar{\sigma}'$. Then it is easy to check that $\bar{\tau}$ satisfies (i). Here note that $\bar{\psi}_\tau \leq (\bar{\varphi} + q_\tau) \circ \bar{\sigma}_\tau$ by Lemma 2.2 (2). So

$$\bar{\psi} \ \leq \ \bar{\psi}_\tau \circ \bar{\sigma}' \ \leq \ ((\bar{\varphi} + q_\tau) \circ \bar{\sigma}_\tau) \circ \bar{\sigma}' \ = \ (\bar{\varphi} + q_\tau) \circ \bar{\tau} \,.$$

But $q_\tau \leq q$ by the choice of q_τ. Therefore $\bar{\tau}$ satisfies (ii). □

Now we can prove Lemma 4.4:

Proof of Lemma 4.4. Define $\rho_+ : H \to F_\sigma\mathsf{ideals}$ as follows: Fix $h \in H$. First, by induction on $n \in \omega$, take ψ_n as $\psi_0 = h(\varphi_{\text{trivial}}, 0)$ and $\psi_{n+1} = h(\psi_n, n)$. Then $\psi = \bigcup_{n\in\omega} \psi_n$ is an unbounded submeasure on ω. Let $\rho_+(h) = \mathsf{Fin}(\psi)$.

Next we shall define $\rho_- : F_\sigma\mathsf{ideals} \to H$. Fix an F_σ-ideal $\mathcal{I}$. Let $\rho_-(\mathcal{I}) \in H$ be as follows: Take a submeasure φ on ω with $\mathcal{I} = \mathsf{Fin}(\varphi)$. Then, by induction on $n \in \omega$, take an increasing sequence $(s_n : n \in \omega)$ in ω so that $s_0 = 0$ and $\varphi(s_{n+1} \setminus s_n) > n$. Let $\varphi_n = \varphi \restriction \mathcal{P}(s_n)$ for each $n \in \omega$. Note that $\varphi_{n+1} \in \Phi(\varphi_n, n)$. Now, for each $(\psi, n) \in \Phi \times \omega$ let $\rho_-(\mathcal{I})(\psi, n) \in \Phi(\psi, n)$ be the one obtained by applying Lemma 4.5 for φ_n, ψ, n and φ_{n+1}. That is, for any $\bar{\psi} \in \Phi(\psi, n)$ with $\rho_-(\mathcal{I})(\psi, n) \trianglelefteq_\psi \bar{\psi}$, any function $\tau : \delta(\psi) \to \delta(\varphi_n)$ and any $q \in \mathbb{Q}_{\geq 0}$ with $\psi \leq (\varphi_n + q) \circ \tau$, there exists a function $\bar{\tau} : \delta(\bar{\psi}) \to \delta(\varphi_{n+1})$ such that

(i) $\bar{\tau}$ extends τ, and $\bar{\tau}[\delta(\bar{\psi}) \setminus \delta(\psi)] \subseteq \delta(\varphi_{n+1}) \setminus \delta(\varphi_n)$,

(ii) $\bar{\psi} \leq (\varphi_{n+1} + q) \circ \bar{\tau}$.

We show that $\rho = (\rho_-, \rho_+)$ is a morphism from $(H, \leq^\circ)$ to $(F_\sigma\mathsf{ideals}, \leq_{\mathrm{KB}})$. Suppose that $\mathcal{I} \in F_\sigma\mathsf{ideals}$, $h \in H$ and $\rho_-(\mathcal{I}) \leq^\circ h$. We must show that $\mathcal{I} \leq_{\mathrm{KB}} \rho_+(h)$.

Let φ and φ_n be as in the definition of $\rho_-(\mathcal{I})$, and let ψ and ψ_n be as in the definition of $\rho_+(h)$. Because $\rho_-(\mathcal{I}) \leq^\circ h$ we can take $m \in \omega$ such that $\rho_-(\mathcal{I})(\psi_n, n) \trianglelefteq_{\psi_n} h(\psi_n, n) = \psi_{n+1}$ for all $n \geq m$. Let $q = \psi_m(\delta(\psi_m))$, and take an arbitrary function $\tau_m : \delta(\psi_m) \to \delta(\varphi_m)$. Note that $\psi_m \leq (\varphi_m + q) \circ \tau_m$. Then, by the construction of $\rho_-(\mathcal{I})$, for each $n \geq m$ we can inductively take $\tau_n : \delta(\psi_n) \to \delta(\varphi_n)$ so that

(i) τ_{n+1} extends τ_n, and $\tau_{n+1}[\delta(\psi_{n+1}) \setminus \delta(\psi_n)] \subseteq \delta(\varphi_{n+1}) \setminus \delta(\varphi_n)$,

(ii) $\psi_{n+1} \leq (\varphi_{n+1} + q) \circ \tau_{n+1}$.

Then $\tau = \bigcup_{n\in\omega\setminus m} \tau_n$ is a finite to one function from ω to ω, and $\psi \leq (\varphi + q) \circ \tau$. Therefore $\mathcal{I} = \mathsf{Fin}(\varphi) \leq_{\mathrm{KB}} \mathsf{Fin}(\psi) = \rho_+(h)$ by Lemma 2.3. □

We have proved Proposition 4.1. Then, as we mentioned before, it suffices for Theorem 2 to prove the following:

Proposition 4.6. $(\omega^\omega, \leq^*) \preceq (\mathsf{summable\ ideals}, F_\sigma\mathsf{ideals}, \leq_{\mathrm{K}})$.

Proof. First we define $\rho_- : \omega^\omega \to \mathsf{summable\ ideals}$. Fix $f \in \omega^\omega$. Take a partition $(A_n : n \in \omega)$ of ω into successive finite intervals such that

(i) $\min A_n \geq f(n)$ for all $n \in \omega \setminus \{0\}$,

(ii) $|A_n|$ is a multiple of 2^n for all $n \in \omega$,

(iii) $|A_n| \geq 2^n \cdot |A_{n'}|$ for all $n', n \in \omega$ with $n' < n$.

Then let $p : \omega \to \mathbb{Q}_{\geq 0}$ be such that $p(k) = 1/|A_n|$ for each $k \in A_n$, and let φ be the measure induced by p. Here note that $\varphi(A_n) = 1$ for each n. So φ is unbounded. Let $\rho_-(f) = \mathsf{Fin}(\varphi) = \{X \subset \omega : \sum_{n \in X} p(n) < \infty\}$.

Next we define $\rho_+ : F_\sigma\mathsf{ideals} \to \omega^\omega$. Suppose that $\mathcal{J} \in F_\sigma\mathsf{ideals}$. Take an unbounded submeasure ψ on ω such that $\mathsf{Fin}(\psi) = \mathcal{J}$. Then define $\rho_+(\mathcal{J}) \in \omega^\omega$ so that

(iv) $\psi(\rho_+(\mathcal{J})(n+1) \setminus \rho_+(\mathcal{J})(n)) \geq n^2(2^n + 1)$.

We show that $\rho = (\rho_-, \rho_+)$ is a morphism from ($\mathsf{summable\ ideals}, F_\sigma\mathsf{ideals}$, $\leq_{\mathrm{K}}$) to $(\omega^\omega, \leq^*)$. Take an arbitrary F_σ-ideal $\mathcal{J}$ with an unbounded submeasure ψ such that $\mathsf{Fin}(\psi) = \mathcal{J}$, and take an arbitrary $f \in \omega^\omega$. Assuming that $f \not\leq^* \rho_+(\mathcal{J})$, we show that $\rho_-(f) \not\leq_{\mathrm{K}} \mathcal{J}$.

For this take an arbitrary function $\tau : \omega \to \omega$. We must find $C \subseteq \omega$ such that $C \in \rho_-(f)$ and $\tau^{-1}[C] \notin \mathcal{J}$. Let $(A_n : n \in \omega)$ and φ be as in the definition of $\rho_-(f)$. It suffices to find $C \subseteq \omega$ such that $\hat{\varphi}(C) < \infty$ and $\hat{\psi}(\tau^{-1}[C]) = \infty$. We may assume that such a finite C does not exists.

First we claim that there are unboundedly many $n \in \omega$ with $\psi(A_n) \geq n(2^n+1)$: Take an arbitrary $m \in \omega$. We will find $n > m$ with $\psi(A_n) \geq n(2^n+1)$. Because $f \not\leq^* \rho_+(\mathcal{J})$, we can take $\bar{n} > m$ such that

$$\min A_{m+1} \ \leq\ \rho_+(\mathcal{J})(\bar{n}) \ <\ \rho_+(\mathcal{J})(\bar{n}+1) \ \leq\ f(\bar{n}+1)\,.$$

Note that $\rho_+(\mathcal{J})(\bar{n}+1) \setminus \rho_+(\mathcal{J})(\bar{n})$ is included in $\bigcup\{A_n : m < n \leq \bar{n}\}$ by (i). Then, because ψ is a submeasure, there is n with $m < n \leq \bar{n}$ such that

$$\begin{aligned} \psi(A_n) \ &\geq\ \frac{\psi(\rho_+(\mathcal{J})(\bar{n}+1) \setminus \rho_+(\mathcal{J})(\bar{n}))}{\bar{n} - m} \ \geq\ \frac{\bar{n}^2(2^{\bar{n}}+1)}{\bar{n} - m} \\ &\geq\ \bar{n}(2^{\bar{n}}+1) \ \geq\ n(2^n+1)\,. \end{aligned}$$

Here the second inequality follows from (iv).

By the claim above we can inductively take $m_i, n_i \in \omega$ for each $i \in \omega$ so that

- $m_i \leq n_i < m_{i+1}$,
- $\hat{\psi}(\tau^{-1}[\bigcup\{A_m : m < m_i\}]) \leq n_i$,
- $\psi(A_{n_i}) \geq n_i(2^{n_i} + 1)$,
- $\tau[A_{n_i}] \subseteq \bigcup\{A_m : m < m_{i+1}\}$.

Let $A'_{n_i} = A_{n_i} \setminus \tau^{-1}[\bigcup\{A_m : m < m_i\}]$. Then we have the following:

(v) $\tau[A'_{n_i}] \subseteq \bigcup\{A_m : m_i \leq m < m_{i+1}\}$.

(vi) $\psi(A'_{n_i}) \geq n_i(2^{n_i} + 1) - n_i = n_i \cdot 2^{n_i}$.

For each $i \in \omega$ we will find $C_i \subseteq \bigcup\{A_m : m_i \le m < m_{i+1}\}$ so that

(vii) $\varphi(C_i) \le \dfrac{1}{2^{m_i-1}}$,

(viii) $\psi(\tau^{-1}[C_i]) \ge n_i$.

Suppose that we could take such C_i's. Let $C = \bigcup_{i\in\omega} C_i$. Then (vii) implies that $\hat{\varphi}(C) < \infty$, and (viii) implies that $\hat{\psi}(\tau^{-1}[C]) = \infty$. So C is as desired.

The construction of C_i's is as follows: Fix $i \in \omega$. Let

$$\begin{array}{ll} M^- = \{m : m_i \le m \le n_i\}\ , & M^+ = \{m : n_i < m < m_{i+1}\}\ , \\ B^- = \bigcup\{A_m : m \in M^-\}\ , & B^+ = \bigcup\{A_m : m \in M^+\}\ , \\ A^- = \tau^{-1}[B^-] \cap A'_{n_i}\ , & A^+ = \tau^{-1}[B^+] \cap A'_{n_i}\ . \end{array}$$

Here note that $A^- \cup A^+ = A'_{n_i}$ by (v). First let $C_i \cap B^+$ be $\tau[A^+]$. Next we define $C_i \cap B^-$. By (ii), for each $m \in M^-$ we can take a partition $(A_{m,j} : j < 2^m)$ of A_m such that $\varphi(A_{m,j}) = |A_{m,j}|/|A_m| = 1/2^m$. For each $j < 2^{n_i}$ let

$$B_j = \bigcup\{A_{m,j} : m \in M^- \,\&\, 2^m > j\}\ .$$

Note that $(B_j : j < 2^{n_i})$ is a partition of B^-. So $A^- \subseteq \bigcup\{\tau^{-1}[B_j] : j < 2^{n_i}\}$, and thus $A'_{n_i} \subseteq \bigcup\{\tau^{-1}[B_j] \cup A^+ : j < 2^{n_i}\}$. Then we can take $j^* < 2^{n_i}$ with

(ix) $\psi(\tau^{-1}[B_{j^*}] \cup A^+) \ge \dfrac{\psi(A'_{n_i})}{2^{n_i}} \ge n_i$.

Let $C_i \cap B^-$ be B_{j^*}.

We must check that C_i satisfies (vii) and (viii). Note that (viii) immediately follows from (ix) and the construction of C_i. We will check (vii). Recall that $\varphi(A_{m,j}) = 1/2^m$ for each $m \in M^-$ and $j < 2^m$. Thus

$$\varphi(C_i \cap B^-) \;=\; \varphi(B_{j^*}) \;\le\; \sum_{m\in M^-} \frac{1}{2^m}\ .$$

Note also that if $m > n_i$, then $|A_{n_i}|/|A_m| \le 1/2^m$ by (iii). So

$$\begin{aligned} \varphi(C_i \cap B^+) \;&=\; \sum_{m\in M^+} \varphi(C_i \cap A_m) \;=\; \sum_{m\in M^+} \frac{|\tau[A^+] \cap A_m|}{|A_m|} \\ &\le\; \sum_{m\in M^+} \frac{|A_{n_i}|}{|A_m|} \;\le\; \sum_{m\in M^+} \frac{1}{2^m}\ . \end{aligned}$$

Then

$$\varphi(C_i) \;\le\; \sum_{m\ge m_i} \frac{1}{2^m} \;=\; \frac{1}{2^{m_i-1}}\ .$$

This completes the proof. $\square$

5 Questions

We end this paper with several questions.

In Section 3 we have proved that $F_\sigma$ideals and Analytic Pideals are upward directed with respect to $\leq_{\mathrm{K}}$ and $\leq_{\mathrm{KB}}$. The first question is on the upward directedness of other families of ideals. Let Borel ideals be the family of all Borel ideals. (Borel ideals, $\leq_{\mathrm{K}}$) is known to be upward directed. However, it seems to be an open problem whether (Borel ideals, $\leq_{\mathrm{KB}}$) is upward directed. (See [4].) It also seems to be an open problem whether (summable ideals, $\leq_{\mathrm{K}}$) and (summable ideals, $\leq_{\mathrm{KB}}$) are upward directed.

Question 5.1. [3]

(1) Is (Borel ideals, $\leq_{\mathrm{KB}}$) *upward directed?*

(2) Are (summable ideals, $\leq_{\mathrm{K}}$) *and* (summable ideals, $\leq_{\mathrm{KB}}$) *upward directed?*

The second question is on the cofinal types of Analytic Pideals and Borel ideals with respect to $\leq_{\mathrm{K}}$ and $\leq_{\mathrm{KB}}$. Recall that $\|(P, \leq_P)\|$ is the least cardinality of a cofinal subfamily of $(P, \leq_P)$ and that $\|(P, \leq_P)^\perp\|$ is that of an unbounded subfamily.

Question 5.2.

(1) How large are $\|$(Analytic Pideals, $\leq_{\mathrm{K}}$)$\|$, $\|$(Analytic Pideals, $\leq_{\mathrm{KB}}$)$\|$, $\|$(Analytic Pideals, $\leq_{\mathrm{K}}$)$^\perp\|$ *and* $\|$(Analytic Pideals, $\leq_{\mathrm{KB}}$)$^\perp\|$*?* [4]

(2) How large are $\|$(Borel ideals, $\leq_{\mathrm{K}}$)$\|$, $\|$(Borel ideals, $\leq_{\mathrm{KB}}$)$\|$, $\|$(Borel ideals, $\leq_{\mathrm{K}}$)$^\perp\|$ *and* $\|$(Borel ideals, $\leq_{\mathrm{KB}}$)$^\perp\|$*?*

Here we make a remark on the above question. It follows from Theorem 2 that summable ideals is unbounded in ($F_\sigma$ideals, $\leq_{\mathrm{K}}$) and ($F_\sigma$ideals, $\leq_{\mathrm{KB}}$). But this is not the case for (Analytic Pideals, $\leq_{\mathrm{K}}$). Let $\mathcal{Z}$ be the density zero ideal, that is,

$$\mathcal{Z} = \left\{ A \subset \omega : \lim_{n\to\infty} \frac{|A \cap n|}{n} = 0 \right\}.$$

$\mathcal{Z}$ is an analytic P-ideal, and the following is known:

Theorem 5.3 (Hernández-Hrušák [3]). *$\mathcal{I} \leq_{\mathrm{K}} \mathcal{Z}$ for all summable ideals $\mathcal{I}$.*

It is known that $\mathcal{Z}$ plays an important role in (Analytic Pideals, $\leq_{\mathrm{K}}$). The following theorem may be useful to investigate $\|$(Analytic Pideals, $\leq_{\mathrm{K}}$)$\|$ and $\|$(Analytic Pideals, $\leq_{\mathrm{K}}$)$^\perp\|$:

[3] This question is positively answered by the second author. See Sakai [7].

[4] It is shown by the second author that there is the largest analytic P-ideal with respect to the Katětov-Blass order. See Sakai [8].

Theorem 5.4 (Measure Dichotomy, Hrušák [4])**.** *Let $\mathcal{I}$ be an analytic P-ideal. Then, either $\mathcal{I} \leq_{\mathrm{K}} \mathcal{Z}$, or there is $X \in \mathcal{I}^+$ such that $\mathcal{S} \leq_{\mathrm{K}} \mathcal{I} \restriction X$, where $\mathcal{S}$ is the ideal on the countable set*

$$\Omega = \left\{ A \in \mathsf{Clop}(2^\omega) : \lambda(A) = \frac{1}{2} \right\}$$

generated by the sets of the form $I_x = \{A \in \Omega : x \in A\}$ for $x \in 2^\omega$. Here λ denotes the standard Haar measure on 2^ω.

The last question is on the existence of a Laflamme family. As we mentioned at the introduction, Laflamme [5] constructs a Laflamme family under CH, and it is not yet known whether the existence of a Laflamme family is provable in ZFC. Actually we can construct a Laflamme family if the pseudo intersection number $\mathfrak{p}$ is greater than or equal to $|F_\sigma \mathsf{ideals}|$, which is equivalent to $\mathfrak{p} = \mathfrak{c}$ because $|F_\sigma \mathsf{ideals}| = \mathfrak{c}$:

Proposition 5.5. *If $\mathfrak{p} = \mathfrak{c}$, then a Laflamme family exists.*

Proof. Assume that $\mathfrak{p} = \mathfrak{c}$. Take an enumeration $(\mathcal{I}_\alpha : \alpha < \mathfrak{c})$ of $F_\sigma \mathsf{ideals}$. By induction on $\alpha < \mathfrak{c}$ we will take an infinite $A_\alpha \subseteq \omega$ so that

- A_α is almost disjoint from A_β for any $\beta < \alpha$,
- $\{A_\beta : \beta \leq \alpha\} \not\subseteq \mathcal{I}_\alpha$,
- $\omega \setminus \bigcup_{\beta \in u} A_\beta$ is infinite for all finite $u \subseteq \alpha + 1$.

Note that if we could construct $\{A_\alpha : \alpha < \mathfrak{c}\}$ as above, then, by extending it to a mad family, we would obtain a Laflamme family. We will use the fact, due to Bell [1], that if $\mathbb{P}$ is a σ-centered poset, and $\mathcal{D}$ is a family of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \mathfrak{p}$, then there is a filter $G \subseteq \mathbb{P}$ intersecting with every element of $\mathcal{D}$.

Suppose that $\alpha < \mathfrak{c}$ and that $(A_\beta : \beta < \alpha)$ has been taken. Let $\mathbb{P}$ be the poset of all pairs (a, u) such that a and u are finite subsets of ω and α, respectively. $(a, u) \leq (b, v)$ in $\mathbb{P}$ if $a \supseteq b$, and $(a \setminus b) \cap A_\beta = \emptyset$ for all $\beta \in v$. Note that $\mathbb{P}$ is σ-centered. For each $\beta < \alpha$ and $n < \omega$ let

$$D^0_\beta = \{(a, u) \in \mathbb{P} : \beta \in u\}\ , \qquad D^1_n = \{(a, u) \in \mathbb{P} : \max a \geq n\}\ .$$

Then D^0_β is clearly dense in $\mathbb{P}$, and so is D^1_n because $\omega \setminus \bigcup_{\alpha \in u} A_\alpha$ is infinite for all finite $u \subseteq \alpha$.

First suppose that $\{A_\beta : \beta < \alpha\} \not\subseteq \mathcal{I}_\alpha$. In this case take a filter $G \subseteq \mathbb{P}$ intersecting with D^0_β and D^1_n for all $\beta < \alpha$ and $n < \omega$. We can take such G because $\alpha < \mathfrak{p}$. Then $A' = \bigcup\{a : \exists u,\ (a, u) \in G\}$ is an infinite subset of ω which is almost disjoint from A_β for every $\beta < \alpha$. Let A_α be an infinite co-infinite subset of A'. Then it is easy to check that A_α is as desired.

Next suppose that $\{A_\beta : \beta < \alpha\} \subseteq \mathcal{I}_\alpha$. Let φ be an unbounded submeasure on ω with $\mathsf{Fin}(\varphi) = \mathcal{I}_\alpha$. Note that $\hat{\varphi}(\omega \setminus \bigcup_{\beta \in u} A_\beta) = \infty$ for any finite $u \subseteq \alpha$ because $\{A_\beta : \beta < \alpha\} \subseteq \mathcal{I}_\alpha$. Then

$$D^2_n = \{(a, u) \in \mathbb{P} : \varphi(a) \geq n\}$$

is dense in $\mathbb{P}$ for all $n < \omega$. Take a filter $G \subseteq \mathbb{P}$ intersecting with D^0_β, D^1_n and D^2_n for all $\beta < \alpha$ and all $n < \omega$, and let $A' = \bigcup\{a : \exists u,\ (a, u) \in G\}$. Then $\hat{\varphi}(A') = \infty$, that is, $A' \notin \mathcal{I}_\alpha$. Because $\mathcal{I}_\alpha$ is an ideal, we can take an infinite co-infinite subset A_α of A' with $A_\alpha \notin \mathcal{I}_\alpha$. Then A_α is as desired. $\square$

Note that $\mathfrak{p} \geq \|(F_\sigma\mathsf{ideals}, \subseteq)\|$ is enough to construct a Laflamme family, because it suffices to take care of a cofinal subfamily of $(F_\sigma\mathsf{ideals}, \subseteq)$ instead of all F_σ-ideals. But this assumption is also equivalent to $\mathfrak{p} = \mathfrak{c}$ by Proposition 1.1. Can we construct a Laflamme family under $\mathfrak{p} \geq \|(F_\sigma\mathsf{ideals}, \leq_{\mathrm{K}})\|$ or $\mathfrak{p} \geq \|(F_\sigma\mathsf{ideals}, \leq_{\mathrm{KB}})\|$? This is equivalent to the following by Corollary 3:

Question 5.6. *Does $\mathfrak{p} = \mathfrak{d}$ imply the existence of a Laflamme family?*

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