Weak diamond principle

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Abstract

Devlin-Shelah [1] introduced the weak diamond principle and proved that it is equivalent with $2^\omega < 2^{\omega_1}$. In this note, we give a proof of this by the argument of Woodin [2] making use of the iteration of ultrapowers.

1 Introduction

In Devlin-Shelah [1], the weak diamond principle was introduced and was proved to be equivalent with $2^\omega < 2^{\omega_1}$:

Definition 1.1 (Devlin-Shelah [1]). *Let* WD*, the weak diamond principle, be the following principle:*

$$\begin{aligned} \mathsf{WD} \ \equiv\ & \forall \mathcal{F} : {}^{<\omega_1}2 \to 2 \ \exists F \in {}^{\omega_1}2 \ \forall G \in {}^{\omega_1}2, \\ & \{\alpha \in \omega_1 \mid \mathcal{F}(G \restriction \alpha) = F(\alpha)\} \text{ is stationary in } \omega_1. \end{aligned}$$

Theorem 1.2 (Devlin-Shelah [1]). $\mathsf{WD} \ \Leftrightarrow\ 2^\omega < 2^{\omega_1}$.

In this note, we present a proof of Theorem 1.2 by the argument of Woodin [2] making use of the iteration of ultrapowers.

This note is constructed as follows. First we review the relation between the ultrapower and the Skolem hull in Section 2. After that, we give a proof of Theorem 1.2 in terms of the iteration of ultrapowers in Section 3.

2 Ultrapower and Skolem hull

Here we review the relation, noticed by Jensen, between ultrapowers and Skolem hulls.

We begin with a review of ultrapowers of models. Let $\mathsf{ZFC}^\triangle$ be the theory

$$\mathsf{ZFC} \ -\ \mathsf{Power\ Set\ Axiom} \ +\ \mathsf{Collection\ Principle}\,.$$

If θ is a regular uncountable cardinal then $\langle \mathcal{H}_\theta, \in \rangle$ is a model of $\mathsf{ZFC}^\triangle$. As is usual, a model $\langle M, \in \rangle$ of $\mathsf{ZFC}^\triangle$ is simply denoted by M.

Suppose that M is a transitive model of $\mathsf{ZFC}^\triangle$ and that $X \in M$ is a nonempty set. Then $\mathcal{P}(X) \cap M$ becomes a Boolean algebra with respect to the operations $\cup$, $\cap$ and $\setminus$. We call an ultrafilter on $\mathcal{P}(X) \cap M$ an *M-ultrafilter over X*. Note that M-ultrafilter need not belong to M.

Let M be a transitive model of $\mathsf{ZFC}^{\triangle}$, X be a nonempty set in M and U be an M-ultrafilter over X. Then let $\mathrm{Ult}(M,U)$ be the usual ultrapower of M by U. That is, $\mathrm{Ult}(M,U) = \langle {}^X M \cap M/\equiv_U , E_U \rangle$ where $\equiv_U$ is the equivalence relation on ${}^X M \cap M$ and E_U is the binary relation on ${}^X M \cap M/\equiv_U$ defined as follows:

- $f \equiv_U g \overset{\text{def}}{\Leftrightarrow} \{x \in X \mid f(x) = g(x)\} \in U$ (for each $f, g \in {}^X M \cap M$)
- $(f)_U \, E_U \, (g)_U \overset{\text{def}}{\Leftrightarrow} \{x \in X \mid f(x) \in g(x)\} \in U$ (for each $f, g \in {}^X M \cap M$)

Here $(f)_U$ denotes the equivalence class in ${}^X M \cap M$ represented by f. It is easy to check that $\equiv_U$ is an equivalence reltation and that E_U is well defined. Moreover it can be shown straightforward that Łoš' theorem holds:

Fact 2.1 (Łoš' theorem)**.** *Suppose that M is a transitive model of $\mathsf{ZFC}^{\triangle}$, that $X \in M$ is a nonempty set and that U is an M-ultrafilter over X. Then for each formula φ and each $f_1, f_2, \ldots, f_n \in {}^X M \cap M$,*

$$\begin{aligned} &\mathrm{Ult}(M,U) \models \varphi[(f_1)_U, (f_2)_U, \ldots (f_n)_U] \\ &\quad \Leftrightarrow \quad \{x \in X \mid M \models \varphi[f_1(x), f_2(x), \ldots, f_n(x)]\} \in U \; . \end{aligned}$$

In particular, $\mathrm{Ult}(M,U)$ becomes a model of $\mathsf{ZFC}^{\triangle}$. Moreover the ultrapower map $i_U : M \to \mathrm{Ult}(M,U)$ is the elementary embedding. Here *the ultrapower map* i_U is the map which maps each $a \in M$ to the equivalence class represented by the constant function with its value a.

Now, further assume that $\mathrm{Ult}(M,U)$ is well founded. Let $\pi : \mathrm{Ult}(M,U) \to \langle N, \in \rangle$ be the transitive collapse and let $j_U := \pi \circ i_U$. Then $j_U : M \to N$ is an elementary embedding. We also call $j_U : M \to N$ *the ultrapower map by U*. Also, for each $f \in {}^X M \cap M$, let $[f]_U$ denote $\pi((f)_U)$.

Now we turn our attension to the relation between ultrapowers and Skolem hulls. The following lemma is the key:

Lemma 2.2. *Suppose that θ is an uncountable regular cardinal and that N is an elementary submodel of $\mathcal{H}_\theta$. Moreover suppose that $X \in N$ is a nonempty set and that $a \in X$. Let $\pi : N \to M$ be the trasitive collapse and let*

$$U \; := \; \{A \in \mathcal{P}(\pi(X)) \cap M \mid a \in \pi^{-1}(A)\} \; .$$

Then U is an M-ultrafilter over $\pi(X)$. Moreover define $k : \mathrm{Ult}(M,U) \to \mathcal{H}_\theta$ as

$$k((f)_U) \; := \; \pi^{-1}(f)(a)$$

for each $f \in {}^{\pi(X)} M \cap M$. Then k is well defined and an elementary embedding.

Proof. It easily follows from the elementarity of π^{-1} that U is an M-ultrafilter. Here we only prove the maximality of U. The others can be proved by the similar argument.

Suppose that $A, B \in \mathcal{P}(\pi(X)) \cap M$ and that $A \cup B = \pi(X)$. Then $\pi^{-1}(A) \cup \pi^{-1}(B) = X$ by the elementarity of π^{-1}. Hence either $a \in \pi^{-1}(A)$ or $a \in \pi^{-1}(B)$. This means that either $A \in U$ or $B \in U$.

Next we show that k is well defined and an elementary embedding. Suppose that φ is an n-ary formula and that $f_1, f_2, \ldots, f_n \in {}^{\pi(X)}M \cap M$. It suffices to show that

$$\begin{aligned} & \mathrm{Ult}(M,U) \models \varphi[(f_1)_U, (f_2)_U, \ldots, (f_n)_U] && (1) \\ \Leftrightarrow\quad & \mathcal{H}_\theta \models \varphi[\pi^{-1}(f_1)(\bar{a}), \pi^{-1}(f_2)(\bar{a}), \ldots, \pi^{-1}(f_n)(\bar{a})] && (2) \end{aligned}$$

(If φ is the formula "$v_1 = v_2$"then this implies well definedness of k).

First, by Łoš' theorem and the definition of U_a,

$$\begin{aligned} (1) \quad \Leftrightarrow \quad & \{x \in \pi(X) \mid M \models \varphi[f_1(x), f_2(x), \ldots, f_n(x)]\} \in U && (3) \\ \Leftrightarrow \quad & a \in \pi^{-1}(\{x \in \pi(X) \mid M \models \varphi[f_1(x), f_2(x), \ldots, f_n(x)]\}) && (4) \end{aligned}$$

But, because π^{-1} is an elementary embedding from M to $\mathcal{H}_\theta$,

$$\begin{aligned} & \pi^{-1}(\{x \in \pi(X) \mid M \models \varphi[f_1(x), f_2(x), \ldots, f_n(x)]\}) \\ & = \{x \in X \mid \mathcal{H}_\theta \models \varphi[\pi^{-1}(f_1)(x), \pi^{-1}(f_2)(x), \ldots, \pi^{-1}(f_n)(x)]\} \end{aligned}$$

Hence (4) $\Leftrightarrow$ (2).

This completes the proof. $\square$

In Lemma 2.2, $\mathrm{Ult}(M,U)$ is embedded into $\mathcal{H}_\theta$. Hence the following holds:

Corollary 2.3. *Suppose that θ, N, X, a, $\pi : N \to M$ and U are as in Lemma 2.2. Then* $\mathrm{Ult}(M,U)$ *is well-founded.*

Lemma 2.2 have the following corollary on the Skolem hull:

Corollary 2.4. *Suppose that θ, M, X and a are as in Lemma 2.2. Let*

$$\bar{N} := \{f(a) \mid f \in N \text{ is a function on } X\}$$

Then $\bar{N}$ is the smallest elementary submodel of $\mathcal{H}_\theta$ including $N \cup \{a\}$.

Proof. Note that $\bar{N}$ is the image of the elementary embedding k in Lemma 2.2. Hence $\bar{N} \prec \mathcal{H}_\theta$. It is clear that $\bar{N}$ is the smallest one. $\square$

Finally we refine Lemma 2.2 slightly. The following corollary gives the framework of our proof of Theorem 1.2:

Corollary 2.5. *Suppose that θ, N, X, a, $\pi : N \to M$ and U be as in Lemma 2.2. Let $\bar{N}$ be as in Cor. 2.4 and let $\bar{\pi} : \bar{N} \to \bar{M}$ be the transitive collapse. Then $\bar{M}$ is also the transitive collapse of* $\mathrm{Ult}(M,U)$. *Let $j_U : M \to \bar{M}$ be the ultrapower map by U. Then the following diagram commutes:*

$$\begin{array}{ccc} N & \xrightarrow{\mathrm{id}} & \bar{N} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\bar{\pi}} \\ M & \xrightarrow[j_U]{} & \bar{M} \end{array}$$

Proof. Let k be as in Lemma 2.2. Then $\bar{\pi} \circ k : \mathrm{Ult}(M,U) \to \bar{M}$ is isomorphic and thus it is the transitive collapse of $\mathrm{Ult}(M,U)$. Here note also that for each $f \in {}^{\pi(X)}M \cap M$,

$$\bar{\pi}^{-1}([f]_U) \;=\; k((f)_U) \;=\; \pi^{-1}(f)(a)\;.$$

To show that the diagram commutes, take an arbitrary $p \in N$. Let $c_{\pi(p)}$ be the constant function on $\pi(X)$ with its value $\pi(p)$ and let c_p be the constant function on X with its value p. Then

$$\bar{\pi}^{-1} \circ j_U \circ \pi(p) \;=\; =\; \bar{\pi}^{-1}([c_{\pi(p)}]_U) \;=\; \pi^{-1}(c_{\pi(p)})(a) \;=\; c_p(a) \;=\; p\;.$$

Therefore $j_U \circ \pi(p) = \bar{\pi}(\bar{p})$.

This completes the proof. □

3 Proof of Theorem 1.2

We give a proof of Theorem 1.2 in terms of the iteration of ultrapowers. We show that the negation of WD is equivalent with $2^\omega = 2^{\omega_1}$. First we transform $\neg$WD slightly:

Lemma 3.1. *The following are equivalent:*

(i) $\neg$WD

(ii) $\exists \mathcal{F} : {}^{<\omega_1}2 \to 2\ \forall F \in {}^{\omega_1}2\ \exists G \in {}^{\omega_1}2,$
$\{\alpha \in \omega_1 \mid \mathcal{F}(G \restriction \alpha) = F(\alpha)\}$ *contains a club in* ω_1.

(iii) $\exists \mathcal{F} : {}^{<\omega_1}2 \to 2\ \forall A \subseteq \omega_1\ \exists G \in {}^{<\omega_1}2,$
$\{\alpha \in \omega_1 \mid \mathcal{F}(G \restriction \alpha) = 1 \Leftrightarrow \alpha \in A\}$ *contains a club in* ω_1.

Proof. (ii) is the statement obtained by restating (iii) using the characteristic function F of A. Hence (ii) and (iii) are equivalent.

We prove the equivalence of (i) and (ii). Note that $\neg$WD is logically equivalent with the following:

$$\exists \mathcal{F} : {}^{<\omega_1}2 \to 2\ \forall F \in {}^{\omega_1}2\ \exists G \in {}^{\omega_1}2,$$
$$\{\alpha \in \omega_1 \mid \mathcal{F}(G \restriction \alpha) = F(\alpha)\} \text{ is nonstationary in } \omega_1.$$

For a function $\mathcal{F} : {}^{<\omega_1}2 \to 2$, let $1-\mathcal{F}$ be the function from ${}^{<\omega_1}2$ to 2 such that $(1-\mathcal{F})(g) = 1 - (\mathcal{F}(g))$ for every $g \in {}^{<\omega_1}2$. Then it is easy to see that if $\mathcal{F}$ witnesses $\neg$WD then $1-\mathcal{F}$ witnesses (ii). On the other hand, if $\mathcal{F}$ witnesses (ii) then $1-\mathcal{F}$ witnesses then $1-\mathcal{F}$ witnesses $\neg$WD. Therefore $\neg$WD and (ii) are equivalent. □

As we mentioned above, we show that $\neg$WD is equivalent with $2^\omega = 2^{\omega_1}$. First we prove the easy direction:

Lemma 3.2. $2^\omega = 2^{\omega_1} \;\Rightarrow\; \neg$WD.

Proof. Assume that $2^\omega = 2^{\omega_1}$. We show that (ii) of Lemma 3.1 holds.

Define $\mathcal{F} : {}^{<\omega_1}2 \to 2$ as follows. First take a surjection $\sigma : {}^{\omega}2 \to {}^{\omega_1}2$. For each $g \in {}^{<\omega_1}2$, let

$$\mathcal{F}(g) := \begin{cases} 0 & \cdots \text{ if } \operatorname{dom} g < \omega \\ \sigma(g \restriction \omega)(\operatorname{dom} g) & \cdots \text{ otherwise} \end{cases}$$

To show that $\mathcal{F}$ witnesses (ii) of Lemma 3.1, take an arbitrary $F \in {}^{\omega_1}2$. Then we can take a $G \in {}^{\omega_1}2$ with $\sigma(G \restriction \omega) = F$. Then for each $\alpha \geq \omega$, $\mathcal{F}(G \restriction \alpha) = F(\alpha)$ by the construction of $\mathcal{F}$. Hence the set $\{\alpha \in \omega_1 \mid \mathcal{F}(G \restriction \alpha) = F(\alpha)\}$ contains a club.

This completes the proof. □

We turn our attention to the other direction:

Lemma 3.3. $\neg\mathsf{WD} \Rightarrow 2^\omega = 2^{\omega_1}$.

The rest of this note is devoted to this lemma. We show that (iii) of Lemma 3.1 implies that $2^\omega = 2^{\omega_1}$. From now, assume that WD fails and fix a witness $\mathcal{F}$ of (iii) of Lemma 3.1. Moreover fix a $\Sigma : \mathcal{P}(\omega_1) \to {}^{\omega_1}2$ such that the set

$$\{\alpha \in \omega_1 \mid \mathcal{F}(\Sigma(A) \restriction \alpha) = 1 \Leftrightarrow \alpha \in A\}$$

contains a club in ω_1 for each $A \in \mathcal{P}(\omega_1)$.

We introduce an iteration of ultrapowers of which is defined from $\mathcal{F}$ (Def. 3.7) and show that every subset of $\mathcal{P}(\omega_1)$ can be taken in the ω_1-th iterate of some countable model of $\mathsf{ZFC}^\triangle$ (Lemma 3.9). As we see later, this easily implies $2^\omega = 2^{\omega_1}$.

Before starting this, we presents a lemma and its corollary which are the core of our argument:

Lemma 3.4. *Suppose that θ is a sufficiently large regular cardinal, that N is a countable elementary submodel of $\mathcal{H}_\theta$ with $\mathcal{F}, \Sigma \in N$. Let $\pi : N \to M$ be the transitive collapse and let $\sigma := \pi(\Sigma)$. Then*

$$\begin{aligned} &\{A \in \mathcal{P}(\omega_1^M) \cap M \mid \mathcal{F}(\sigma(A)) = 1\} \\ &\quad = \{A \in \mathcal{P}(\omega_1^M) \cap M \mid N \cap \omega_1 \in \pi^{-1}(A)\} . \end{aligned}$$

Proof. Take an arbitrary $A \in \mathcal{P}(\omega_1^M) \cap M$. Then, by the elementarity of N, there is a club $C \subseteq \omega_1$ in N such that

$$\mathcal{F}(\Sigma(\pi^{-1}(A)) \restriction \alpha) = 1 \Leftrightarrow \alpha \in \pi^{-1}(A)$$

for every $\alpha \in C$. Note that $N \cap \omega_1 \in C$ because $C \in N$ is a club in ω_1. Note also that $\Sigma(\pi^{-1}(A)) \restriction (N \cap \omega_1) = \sigma(A)$. Hence $\mathcal{F}(\sigma(A)) = 1$ if and only if $N \cap \omega_1 \in \pi^{-1}(A)$.

This implies the lemma. □

Corollary 3.5. *Suppose that θ is a sufficiently large regular cadinal, that N is a countable elementary submodel of $\mathcal{H}_\theta$ with $\mathcal{F}, \Sigma \in N$. Let $\pi : N \to M$ be the transitive collapse and let $\sigma := \pi(\Sigma)$. Then*

$$W \;:=\; \{A \in \mathcal{P}(\omega_1^M) \cap M \mid \mathcal{F}(\sigma(A)) = 1\}$$

is an M-ultrafilter over ω_1^M. Moreover let

$$\bar{N} \;:=\; \{f(N \cap \omega_1) \mid f \in N \text{ is a function on } \omega_1\}$$

and let $\bar{\pi} : \bar{N} \to \bar{M}$ be the transitive collapse. Then $\bar{M}$ is isomorphic to $\mathrm{Ult}(M, W)$. *Futhermore if we let $j_W : M \to \bar{M}$ be the ultrapower map then the following diagram commutes:*

$$\begin{array}{ccc} N & \xrightarrow{\mathrm{id}} & \bar{N} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\bar{\pi}} \\ M & \xrightarrow[j_W]{} & \bar{M} \end{array}$$

An important thing in the above corollary is that, referring to the fixed $\mathcal{F}$, W can be defined by σ which is hereditarily countable. This corollary invoke the following definition of the $\mathcal{F}$-iteration of ultrapowers:

Definition 3.6. *We call a pair $\langle M, \sigma\rangle$ with the following properties a $\mathcal{F}$-mouse:*

(1) *M is a countable transitive model of $\mathsf{ZFC}^{\triangle}$ and $M \models$ "ω_1 exists".*

(2) *$\sigma \in M$ and $M \models$ "σ is a function from $\mathcal{P}(\omega_1)$ to ${}^{\omega_1}2$".*

(3) *$W_\sigma := \{A \in \mathcal{P}(\omega_1^M) \cap M \mid \mathcal{F}(\sigma(A)) = 1\}$ is an M-ultrafilter over ω_1^M.*

Definition 3.7. *Let $\langle M, \sigma\rangle$ be a $\mathcal{F}$-mouse. Then $\langle M_\alpha, \sigma_\alpha, j_{\alpha\beta} \mid \alpha \le \beta < \omega_1\rangle$ is called an $\mathcal{F}$-iteration of ultrapowers of $\langle M, \sigma\rangle$ if it satisfies the following:*

(1) *$\langle M_0, \sigma_0\rangle = \langle M, \sigma\rangle$ and $\langle M_\alpha, \sigma\rangle$ is an $\mathcal{F}$-mouse for each α.*

(2) *$j_{\alpha\beta} : M_\alpha \to M_\beta$ is an elementary embedding for each α, β and $\langle M_\alpha, j_{\alpha\beta} \mid \alpha \le \beta \le \omega_1\rangle$ is a directed system.*

(3) *$\sigma_\alpha = j_{0\alpha}(\sigma)$.*

(4) *If γ is a countable limit ordinal then M_γ is isomorphic to the direct limit of $\langle M_\alpha, j_{\alpha\beta} \mid \alpha \le \beta < \gamma\rangle$ and, for each $\alpha < \gamma$, $j_{\alpha\gamma} : M_\alpha \to M_\gamma$ is the map naturally induced from the direct limit map.*

(5) *$M_{\alpha+1}$ is isomorphic to* $\mathrm{Ult}(M_\alpha, W_{\sigma_\alpha})$ *and $j_{\alpha,\alpha+1} : M_\alpha \to M_{\alpha+1}$ is the ultrapower map by W_{σ_α} for each α.*

Note 3.8. *Suppose that $\langle M, \sigma\rangle$ is an $\mathcal{F}$-mouse. Then an $\mathcal{F}$-iteration of $\langle M, \sigma\rangle$ may not be exists. But if exists then it is unique.*

Lemma 3.9. *Suppose that $P \in \mathcal{P}(\omega_1)$. Then there is a $\mathcal{F}$-mouse $\langle M, \sigma \rangle$ and the $\mathcal{F}$-iteration $\langle M_\alpha, \sigma_\alpha, j_{\alpha\beta} \mid \alpha \leq \beta < \omega_1 \rangle$ of ultrapowers of $\langle M, \sigma \rangle$ such that P is in the transitive collapse of the direct limit of $\langle M_\alpha, j_{\alpha\beta} \mid \alpha \leq \beta < \omega_1 \rangle$.*

Proof. Take an sufficiently large regular cardinal θ and an countable elementary submodel N of $\mathcal{H}_\theta$ with $\mathcal{F}, \Sigma, P \in N$. Then by induction on $\alpha < \omega_1$, define $\langle N_\alpha \mid \alpha < \omega_1 \rangle$ as follows:

- $N_0 := N$.
- $N_{\alpha+1} := \{ f(N_\alpha \cap \omega_1) \mid f \in N \text{ is a function on } \omega_1 \}$ for each $\alpha < \omega_1$.
- $N_\gamma = \bigcup_{\alpha<\gamma} N_\alpha$ for each limit $\gamma < \omega_1$.

By Cor. 2.4, $\langle N_\alpha \mid \alpha < \omega_1 \rangle$ is a continuous $\subseteq$-increasing sequence of elementary submodels of $\mathcal{H}_\theta$.

For each $\alpha < \omega_1$, let $\pi_\alpha : N_\alpha \to M_\alpha$ be the transitive collapse and let $\sigma_\alpha := \pi_\alpha(\Sigma)$. Moreover for each $\alpha, \beta < \omega_1$ with $\alpha \leq \beta$, let $j_{\alpha\beta} : M_\alpha \to M_\beta$ be $\pi_\beta \circ \pi_\alpha^{-1}$.

Then, by Cor. 3.5, it can be easily checked that $\langle M_\alpha, \sigma_\alpha, j_{\alpha\beta} \mid \alpha \leq \beta < \omega_1 \rangle$ is an $\mathcal{F}$-iteration of $\langle M_0, \sigma_0 \rangle$.

Let $N^* := \bigcup_{\alpha<\omega_1} N_\alpha$ and let $\pi^* : N^* \to M^*$ be the transitive collapse. Then it is easy to see that M^* is also the transitive collapse of the direct limit of $\langle M_\alpha, j_{\alpha\beta} \mid \alpha \leq \beta < \omega_1 \rangle$. Moreover $P = \pi^*(P) \in M^*$ because $\omega_1 \subseteq N^*$.

This completes the proof. □

From Note 3.8 and Lemma 3.9, we can deduce $2^\omega = 2^{\omega_1}$ easily:

First note that there are at most 2^ω many $\mathcal{F}$-mice. Next, for each $\mathcal{F}$-mouse, there exists at most one $\mathcal{F}$-iteration of ultrapowers of it. Finally, for each $\mathcal{F}$-iteration of ultrapowers, the transitive collapse of the direct limit of it has the size ω_1. Hence, by Lemma 3.9,

$$2^{\omega_1} \ \leq \ 2^\omega \times 1 \times \omega_1 \ = \ 2^\omega \ .$$

Therefore $2^\omega = 2^{\omega_1}$.

Now we have proved Lemma 3.3 and this completes the proof of Theorem 1.2.

References

[1] K. J. Devlin and S. Shelah, *A weak version of $\diamondsuit$ which follows from $2^{\aleph_0} < 2^{\aleph_1}$*, Israel J. of Math. **29** (1978), no.2-3, 239-247.

[2] H. Woodin, *The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal*, De Gruyter Series in Logic and its Applications; 1, Walter de Gruyter, Berlin, New York, 1999.