# $\mathsf{TP}$ and weak square

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**Notation 1.** *For a regular cardinal $\kappa \geq \omega_2$, an ordinal $\lambda \geq \omega_2$ and an ordinal $\xi$ let $\mathrm{Fn}(\kappa, \lambda, \xi)$ denote the set of all partial functions $f : \lambda \to \xi$ with $|\mathrm{dom}(f)| < \kappa$.*

**Definition 2.** *Let $\kappa$ be a regular cardinal $\geq \omega_2$, $\lambda$ be an ordinal $\geq \kappa$ and $\xi$ be an ordinal.*

*(1) An $(\kappa, \lambda, \xi)$*-tree *is a family $T \subseteq \mathrm{Fn}(\kappa, \lambda, \xi)$ such that*

- *$f \restriction u \in T$ for any $f \in T$ and any $u \subseteq \mathrm{dom}(f)$,*
- *$T_u := \{f \in T \mid \mathrm{dom}(f) = u\} \neq \emptyset$ for all $u \in [\lambda]^{<\kappa}$.*

*(2) A $(\kappa, \lambda, \xi)$-tree $T$ is said to be* thin *if $|T_u| < \kappa$ for all $u \in [\lambda]^{<\kappa}$.*

*(3) A total function $F : \lambda \to \xi$ is said to be a* cofinal branch *of a $(\kappa, \lambda, \xi)$-tree $T$ if $F \restriction u \in T$ for all $u \in [\lambda]^{<\kappa}$.*

*(4) $\mathsf{TP}(\kappa, \lambda)$ denotes the statement that every thin $(\kappa, \lambda, 2)$-tree has a cofinal branch.*

**Remark 3.** *It is easy to see that $\mathsf{TP}(\kappa, \lambda)$ is equivalent to that every thin $(\kappa, \lambda, \omega)$-tree has a cofinal branch.*

**Definition 4.** *Let $\kappa$ be a cardinal and $\lambda$ be a regular cardinal $\geq \kappa$.*

$\square(\lambda, {<}\kappa) \;\equiv\;$ *There is a sequence $\langle \mathcal{C}_\alpha \mid \alpha < \lambda \rangle$ with the following properties:*

*(i) Each $\mathcal{C}_\alpha$ is a non-empty family of club subsets of $\alpha$ such that $|\mathcal{C}_\alpha| < \kappa$.*

*(ii) If $c \in \mathcal{C}_\alpha$, and $\beta \in \mathrm{Lim}(c)$, then $c \cap \beta \in \mathcal{C}_\beta$.*

*(iii) There are no club $C \subseteq \lambda$ such that $C \cap \alpha \in \mathcal{C}_\alpha$ for all $\alpha \in \mathrm{Lim}(C)$.*

*A sequence $\langle \mathcal{C}_\alpha \mid \alpha < \lambda \rangle$ with the properties (i)–(iii) above is called a $\square(\lambda, {<}\kappa)$-*sequence*.*

We prove the following:

**Theorem 5.** *Let $\kappa$ and $\lambda$ be regular cardinals with $\omega_2 \leq \kappa \leq \lambda$. Then $\mathsf{TP}(\kappa, \lambda)$ implies that $\square(\lambda, {<}\kappa)$ fails.*

The rest of this note is devoted to the proof of Thm.5.

Fix regular cardinals $\kappa$ and $\lambda$ with $\omega_2 \leq \kappa \leq \lambda$, and assume that $\square(\lambda, <\kappa)$. We prove that $\mathsf{TP}(\kappa, \lambda)$ fails.

Let $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \lambda \rangle$ be a $\square(\lambda, <\kappa)$-sequence. We may assume that $\mathcal{C}_\alpha = \{\{\alpha - 1\}\}$ for all successor $\alpha < \lambda$. For each $\alpha < \lambda$ choose $c_\alpha \in \mathcal{C}_\alpha$ arbitrarily.

We use minimal walks through $\langle c_\alpha \mid \alpha < \lambda \rangle$. For each $\alpha, \beta$ with $\beta < \alpha < \lambda$, taking a finite decreasing sequence $\langle \alpha_0, \alpha_1, \ldots, \alpha_n \rangle$ such that

- $\alpha_0 = \alpha$, and $\alpha_n = \beta$,
- $\alpha_{k+1} = \min(c_{\alpha_k} \setminus \beta)$ for each $k$,

let

$$\begin{aligned} w(\alpha, \beta) &:= \langle \alpha_0, \alpha_1, \ldots, \alpha_n \rangle \ , \\ d(\alpha, \beta) &:= n \ . \end{aligned}$$

For each $\alpha < \lambda$ let $d_\alpha : \alpha \to \omega$ be the function such that $d_\alpha(\beta) = d(\alpha, \beta)$.

We show that

$$T := \{ d_\alpha \restriction u \mid u \in [\lambda]^{<\kappa} \wedge \alpha \geq \overline{\sup}(u) \}$$

is a thin $(\kappa, \lambda, \omega)$-tree without cofinal branches. Clearly $T$ is a $(\kappa, \lambda, \omega)$-tree. Thus it suffices to prove that $T$ is thin and that $T$ has no cofinal branches.

To prove these we need some preparations. For each $\alpha, \beta$ with $\beta < \alpha < \lambda$ and each $c \in \mathcal{C}_\alpha$, taking a finite decreasing sequence $\langle \alpha_0, \alpha_1, \ldots, \alpha_n \rangle$ such that

- $\alpha_0 = \alpha$, and $\alpha_n = \beta$,
- $\alpha_1 = \min(c \setminus \beta)$,
- $\alpha_{k+1} = \min(c_{\alpha_k} \setminus \beta)$ for each $k \geq 1$,

let

$$\begin{aligned} w^c(\alpha, \beta) &:= \langle \alpha_0, \alpha_1, \ldots, \alpha_n \rangle \ , \\ d^c(\alpha, \beta) &:= n \ . \end{aligned}$$

(Thus we use $c$ in the first step instead of $c_\alpha$.) For each $c \in \mathcal{C}_\alpha$ let $d^c_\alpha : \alpha \to \omega$ be the function such that $d^c_\alpha(\beta) = d^c(\alpha, \beta)$.

**Remark 6.** *$c = \{\beta < \alpha \mid d^c_\alpha(\beta) = 1\}$ for each $\alpha < \lambda$ and each $c \in \mathcal{C}_\alpha$.*

**Lemma 7.** *Suppose that $\alpha \leq \alpha' < \lambda$ and that $\alpha$ is a limit ordinal. Then there exist $\gamma < \alpha$, $c \in \mathcal{C}_\alpha$ and $m \in \omega$ such that $d_{\alpha'}(\beta) = d^c_\alpha(\beta) + m$ for all $\beta \in \alpha \setminus \gamma$.*

*Proof.* If $\alpha = \alpha'$, then the lemma is clear. Assume that $\alpha < \alpha'$. Let $w(\alpha', \alpha)$ be $\langle \alpha'_0, \alpha'_1, \ldots, \alpha'_m \rangle$. The proof splits into two cases.

First suppose that $c_{\alpha'_{m-1}} \cap \alpha$ is not unbounded in $\alpha$. In this case let

$$\gamma := \max\{\max(c_{\alpha'_k} \cap \alpha) + 1 \mid k \leq m - 1\} \ .$$

Then $\gamma < \alpha$. Moreover it is easy to see that for any $\beta \in \alpha \setminus \gamma$ the walk $w(\alpha', \beta)$ goes through $\alpha$. More precisely, if $\beta \in \alpha \setminus \gamma$, and $w(\alpha, \beta) = \langle \alpha_0, \alpha_1, \ldots, \alpha_n \rangle$, then

$$w(\alpha', \beta) = \langle \alpha'_0, \alpha'_1, \ldots, \alpha'_m = \alpha = \alpha_0, \alpha_1, \ldots, \alpha_n \rangle \ .$$

Therefore $d_{\alpha'}(\beta) = d_\alpha^{c_\alpha}(\beta) + m$ for all $\beta \in \alpha \setminus \gamma$.

Next suppose that $c_{\alpha'_{m-1}} \cap \alpha$ is unbounded in $\alpha$. In this case let

$$\gamma := \max\{\max(c_{\alpha'_k} \cap \alpha) + 1 \mid k < m-1\} \ .$$

Then $\gamma < \alpha$. Note that $c := c_{\alpha'_{m-1}} \cap \alpha \in \mathcal{C}_\alpha$ by the coherency. We prove that $d_{\alpha'}(\beta) = d^c_\alpha(\beta) + m - 1$ for all $\beta \in \alpha \setminus \gamma$.

Suppose that $\beta \in \alpha \setminus \gamma$. Let $w(\alpha'_{m-1}, \beta) = \langle \alpha_0, \alpha_1, \ldots, \alpha_n \rangle$. Then

$$w(\alpha', \beta) = \langle \alpha'_0, \alpha'_1, \ldots, \alpha'_{m-1} = \alpha_0, \alpha_1, \ldots, \alpha_n \rangle \ .$$

Moreover

$$w^c(\alpha, \beta) = \langle \alpha, \alpha_1, \ldots, \alpha_n \rangle$$

because $c_{\alpha'_{m-1}} \cap \alpha = c$. Therefore $d_{\alpha'}(\beta) = d^c_\alpha(\beta) + m - 1$. □

Using Lem.7 we can easily prove that $T$ is thin:

*Proof of that $T$ is thin.* By induction on $\overline{\sup}(u)$ we prove that $|T_u| < \kappa$ for every $u \in [\lambda]^{<\kappa}$. If $\overline{\sup}(u) = 0$, i.e. $u = \emptyset$, then this is clear.

Suppose that $\alpha < \lambda$ and that $|T_u| < \kappa$ for every $u \in [\lambda]^{<\kappa}$ with $\overline{\sup}(u) < \alpha$. If $\alpha$ is a successor ordinal, then for every $u \in [\lambda]^{<\kappa}$ with $\overline{\sup}(u) = \alpha$ we have that

$$|T_u| \ \leq \ |T_{u \cap \alpha - 1} \times \omega| \ < \ \kappa$$

because $T_u$ consists of functions from $u$ to $\omega$. Suppose that $\alpha$ is a limit ordinal. Take an arbitrary $u \in [\lambda]^{<\kappa}$ with $\overline{\sup}(u) = \alpha$. Here recall that $T_u = \{d_{\alpha'} \restriction u \mid \alpha' \geq \alpha\}$. By Lem.7, for each $\alpha' \geq \alpha$ we can take $\gamma < \alpha$, $c \in \mathcal{C}_\alpha$ and $m \in \omega$ such that $d_{\alpha'}(\beta) = d^c_\alpha(\beta) + m$ for every $\beta \in \alpha \setminus \gamma$. Thus

$$|T_u| \ \leq \ \left|\left(\bigcup_{\gamma<\alpha} T_{u\cap\gamma}\right) \times \mathcal{C}_\alpha \times \omega\right| \ < \ \kappa \ .$$

□

Finally we prove that $T$ has no cofinal branches:

*Proof of that $T$ has no cofinal branches.* For the contradiction assume that $T$ has a cofinal branch $F$. We will construct a club $C \subseteq \lambda$ such that $C \cap \alpha \in \mathcal{C}_\alpha$ for every $\alpha \in \mathrm{Lim}(C)$. This will contradict that $\vec{\mathcal{C}}$ is a $\square(\lambda, {<}\kappa)$-sequence.

For each $m \in \omega$ let

$$C_m \ := \ \{\beta < \lambda \mid F(\beta) = m\} \ .$$

We claim the following:

**Claim.** *For each $\alpha \in E^{\lambda}_{<\kappa}$ there exist $\gamma < \alpha$, $c \in \mathcal{C}_\alpha$ and $m \in \omega$ such that $(C_m \cap \alpha) \setminus \gamma = c \setminus \gamma$.*

*Proof of Claim.* Fix $\alpha \in E^{\lambda}_{<\kappa}$. Take an unbounded $b \subseteq \alpha$ of order-type $\mathrm{cf}(\alpha)$.

Because $F$ is a cofinal branch of $T$, for each $u \in [\alpha]^{<\kappa}$ we can take $\alpha_u \in \lambda \setminus \alpha$ such that $F \restriction u = d_{\alpha_u} \restriction u$. Then by Rmk.6 and Lem.7, for each $u \in [\alpha]^{<\kappa}$ with $\overline{\sup}(u) = \alpha$, we can take $\gamma_u \in b$, $c_u \in \mathcal{C}_\alpha$ and $m_u \in \omega$ such that

$$(C_{m_u} \cap u) \setminus \gamma_u \;=\; (c_u \cap u) \setminus \gamma_u \ .$$

Then we can take $\gamma \in b$, $c \in \mathcal{C}_\alpha$ and $m \in \omega$ such that there are $\subseteq$-cofinally many $u \in [\alpha]^{<\kappa}$ with $(\gamma_u, c_u, m_u) = (\gamma, c, m)$. Then $\gamma$, $c$ and $m$ witness the claim clearly. $\square_{\text{Claim}}$

For each $\alpha \in E^{\lambda}_{<\kappa}$ take $\gamma_\alpha < \alpha$, $c^*_\alpha \in \mathcal{C}_\alpha$ and $m_\alpha \in \omega$ witnessing the claim above. By Fodor's lemma we can take $\gamma < \lambda$ and $m \in \omega$ such that the set $B_0 := \{\alpha \in E^{\lambda}_{<\kappa} \mid (\gamma_\alpha, m_\alpha) = (\gamma, m)\}$ is stationary in $\lambda$. Here note that $|\{c^*_\alpha \cap \gamma \mid \alpha^* \in B_0\}| < \lambda$ by the coherency of $\vec{\mathcal{C}}$. Therefore we can take $c^* \subseteq \gamma$ such that the set $B_1 := \{\alpha \in B_0 \mid c^*_\alpha \cap \gamma = c^*\}$ is stationary.

Let

$$C \;:=\; (C_m \setminus \gamma) \cup c^* \ .$$

Then $C \cap \alpha = c^*_\alpha$ for every $\alpha \in B_1$ by the construction of $B_1$. Using this fact, it is easy to see that $C$ is club in $\lambda$ and that $C \cap \alpha \in \mathcal{C}_\alpha$ for every $\alpha \in \mathrm{Lim}(C)$. This contradicts that $\vec{\mathcal{C}}$ is a $\square(\lambda, {<}\kappa)$-sequence. $\square$

This completes the proof of Thm.5.