# $\omega_1$-stationary preserving poset of size $\omega_1$ which is not semi-proper

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## 1 Introduction

In this note, we discuss the existence of an $\omega_1$-stationary preserving poset of size $\omega_1$ which is not semi-proper. First we will prove that the following combinatorial principle $\diamondsuit^{++}$, which strengthens $\diamondsuit^{+}$, implies the existence of such a poset.

**Notation 1.1.** *For a set $M$ and a well-founded relation $E$ on $M$ let $\mathrm{tcol}(M,E)$ denote the transitive collapse of $\langle M, E\rangle$. If $E = \in \cap\, {}^2M$, then we simply write $\mathrm{tcol}(M)$ for $\mathrm{tcol}(M,E)$.*

**Definition 1.2.** *Let $\diamondsuit^{++}$ be the following combinatorial principle:*

$\diamondsuit^{++}$ $\equiv$ *There exists a sequence $\vec{K} = \langle K_\alpha \mid \alpha < \omega_1\rangle$ with the following properties:*

*(i) Each $K_\alpha$ is a countable set.*

*(ii) For any $A \subseteq \omega_1$ there is a club $C \subseteq \omega_1$ such that both $A \cap \alpha$ and $C \cap \alpha$ belong to $K_\alpha$ for each $\alpha \in C$.*

*(iii) The set*

$$X_{\vec{K}} := \{M \in [\mathcal{H}_{\omega_2}]^\omega \mid M \cap \omega_1 \in \omega_1 \wedge \mathrm{tcol}(M) = K_{M\cap\omega_1}\}$$

*is stationary in $[\mathcal{H}_{\omega_2}]^\omega$.*

*A sequence $\vec{K} = \langle K_\alpha \mid \alpha < \omega_1\rangle$ satisfying the properties (i)–(iii) above is called a $\diamondsuit^{++}$-sequence.*

**Proposition 1.3.** *If $\diamondsuit^{++}$ holds, then there exists an $\omega_1$-stationary preserving poset of size $\omega_1$ which is not semi-proper.*

Prop.1.3 will be proved in Section 2. As for the consistency of $\diamondsuit^{++}$ we will prove the following in Section 3:

**Proposition 1.4.** *$\diamondsuit^{++}$ holds in the constructible universe $L$.*

**Proposition 1.5.** *There exists a poset forcing $\diamondsuit^{++}$.*

Thus we will have the following corollaries:

**Corollary 1.6.** *In $L$ there exists an $\omega_1$-stationary preserving poset of size $\omega_1$ which is not semi-proper.*

**Corollary 1.7.** *There exists a poset forcing the existence of an $\omega_1$-stationary preserving poset of size $\omega_1$ which is not semi-proper.*

## 2 $\diamondsuit^{++}$ and $\omega_1$-stationary preserving poset of size $\omega_1$ which is not semi-proper

In this section we prove Prop.1.3. Note that a poset with the $< \omega_2$-c.c. is semi-proper if and only if it is proper. Thus, it suffices to prove that if $\diamondsuit^{++}$ holds, then there exists an $\omega_1$-stationary preserving poset of size $\omega_1$ which is not proper. In fact, we will prove the following.

**Proposition 2.1.** *If $\diamondsuit^{++}$ implies holds, then there exists an $\omega_1$-stationary preserving poset of size $\omega_1$ which does not preserve stationary subsets of $[\mathcal{H}_{\omega_2}]^\omega$.*

First we slightly improve $\diamondsuit^{++}$:

**Notation 2.2.** *Let $\mathsf{ZFC}^-$ denote the axiom system $\mathsf{ZFC} - \mathsf{Power\ Set\ Axiom}$.*

**Lemma 2.3.** *Assume $\diamondsuit^{++}$. Then there is a $\diamondsuit^{++}$-sequence $\vec{K} = \langle K_\alpha \mid \alpha < \omega_1 \rangle$ with the following properties:*

*(iv) $\vec{K}$ is $\subseteq$-increasing.*

*(v) Each $K_\alpha$ is a countable transitive model of $\mathsf{ZFC}^-$ with $\vec{K} \restriction \alpha \in K_\alpha$.*

*Proof.* Let $\vec{K}' = \langle K'_\alpha \mid \alpha < \omega_1 \rangle$ be a $\diamondsuit^{++}$-sequence. By induction on $\alpha < \omega_1$ define $\vec{K} = \langle K_\alpha \mid \alpha < \omega_1 \rangle$ as follows: If $K'_\alpha$ is a countable transitive model of $\mathsf{ZFC}^-$ with $\bigcup_{\beta<\alpha} K_\beta \subseteq K'_\alpha$ and $\vec{K} \restriction \alpha \in K'_\alpha$, then let $K_\alpha := K'_\alpha$. Otherwise let $K_\alpha$ be an arbitrary countable transitive model of $\mathsf{ZFC}^-$ with $(\bigcup_{\beta<\alpha} K_\beta) \cup K'_\alpha \subseteq K_\alpha$ and $\vec{K} \restriction \alpha \in K_\alpha$.

Then $\vec{K}$ satisfies the properties (iv) and (v) in Lem.2.3. Thus it suffices to show that $\vec{K}$ is a $\diamondsuit^{++}$-sequence. Clearly $\vec{K}$ satisfies the properties (i) and (ii) in the definition of $\diamondsuit^{++}$ (Def.1.2). Below we check (iii).

Take an arbitrary function $F : [\mathcal{H}_{\omega_2}]^{<\omega} \to \mathcal{H}_{\omega_2}$. We must find $M \in X_{\vec{K}}$ which is closed under $F$. Because $\vec{K}'$ is a $\diamondsuit^{++}$-sequence, we can take $M \in X_{\vec{K}'}$ with $M \prec \langle \mathcal{H}_{\omega_2}, \in, \vec{K}, F \rangle$. Then $M$ is closed under $F$. We show that $M \in X_{\vec{K}}$. Let $\alpha := M \cap \omega_1$. It suffices to show that $\mathrm{tcol}(M) = K_\alpha$. First note that $K'_\alpha = \mathrm{tcol}(M)$ because $M \in X_{\vec{K}'}$. Thus $K'_\alpha$ is a countable transitive model of $\mathsf{ZFC}^-$. Note also that if we let $\tau : M \to K'_\alpha$ be the transitive collapse, then $\vec{K} \restriction \alpha = \tau(\vec{K})$. Hence $\bigcup_{\beta<\alpha} K_\beta \subseteq K'_\alpha$, and $\vec{K} \restriction \alpha \in K'_\alpha$. So $K_\alpha = K'_\alpha$. Therefore $\mathrm{tcol}(M) = K'_\alpha = K_\alpha$. □

Now we present a poset $\mathbb{P}_{\vec{K}}$ which witnesses Prop.2.1:

**Definition 2.4.** *Let $\vec{K} = \langle K_\alpha \mid \alpha < \omega_1 \rangle$ be a $\diamondsuit^{++}$-sequence satisfying the properties (iv) and (v) in Lem.2.3. Then let $\mathbb{P}_{\vec{K}}$ be the poset of all $p \in {}^{<\omega_1}2$ with $p \in K_{\mathrm{dom}(p)}$. $p \leq q$ in $\mathbb{P}_{\vec{K}}$ if $p \supseteq q$.*

Note that $\diamondsuit^{++}$ implies $\mathsf{CH}$. Thus $\mathbb{P}_{\vec{K}}$ has the size $\omega_1$. So it suffices for Prop.2.1 to prove the following:

**Lemma 2.5.** *Suppose that $\vec{K} = \langle K_\alpha \mid \alpha < \omega_1 \rangle$ is a $\diamondsuit^{++}$-sequence satisfying the properties (iv) and (v) in Lem.2.3.*

*(1) $\mathbb{P}_{\vec{K}}$ is $\omega_1$-stationary preserving.*

*(2) $X_{\vec{K}}$ in Def.1.2 defined in $V$ becomes non-stationary in $V^{\mathbb{P}_{\vec{K}}}$.*

Before proving Lem.2.5, note the following:

**Lemma 2.6.** *Suppose that $\vec{K} = \langle K_\alpha \mid \alpha < \omega_1 \rangle$ is a $\diamondsuit^{++}$-sequence satisfying the properties (iv) and (v) in Lem.2.3 and that $G$ is $\mathbb{P}_{\vec{K}}$-generic filter over $V$. Then $\mathrm{dom}(\bigcup G) = (\omega_1)^V$.*

*Proof.* In $V$ it suffices to show that $\{q \in \mathbb{P}_{\vec{K}} \mid \mathrm{dom}(q) \geq \alpha\}$ is dense. Take an arbitrary $p \in \mathbb{P}_{\vec{K}}$ and an arbitrary $\alpha \in \omega_1 \setminus \mathrm{dom}(p)$. Let $q$ be the function on $\alpha+1$ such that $q \restriction \mathrm{dom}(p) = p$ and such that $q(\beta) = 0$ for all $\beta \in \alpha+1 \setminus \mathrm{dom}(p)$. Then $q \leq p$, and $\mathrm{dom}(q) \geq \alpha$. □

Now we prove Lem.2.5:

*Proof of Lem.2.5.* (1) We work in $V$. Suppose that $S$ is a stationary subset of $\omega_1$, that $p \in \mathbb{P}_{\vec{K}}$ and that $\dot{C}$ is a $\mathbb{P}_{\vec{K}}$-name for a club subset of $\omega_1$. It suffices to find $p^* \leq p$ and $\alpha^* \in S$ such that $p^* \Vdash$ " $\alpha \in \dot{C}$ ".

Take a well-order $\Delta$ of $\mathbb{P}_{\vec{K}}$. Moreover for each $\alpha < \omega_1$ let

$$\begin{aligned} \mathbb{P}_{<\alpha} &:= \mathbb{P}_{\vec{K}} \cap {}^{<\alpha}2 \,, \\ D_\alpha &:= \{q \in \mathbb{P}_{\vec{K}} \mid q \text{ decides } \min(\dot{C} \setminus \alpha)\} \,. \end{aligned}$$

Note that $\mathbb{P}_{<\alpha} \in K_\alpha$ by the construction of $\mathbb{P}_{\vec{K}}$ and the properties (iv) and (v) in Lem.2.3 of $\vec{K}$. Note also that each $D_\alpha$ is a dense open subset of $\mathbb{P}_{\vec{K}}$.

Because $\vec{K}$ is a $\diamondsuit^+$-sequence, we can easily take a club $B \subseteq \omega_1$ such that

$$p\,,\ \Delta \cap \mathbb{P}_{<\alpha}\,,\ \langle D_\beta \cap \mathbb{P}_{<\alpha} \mid \beta < \alpha\rangle\,,\ B \cap \alpha \ \in\ K_\alpha$$

for each $\alpha \in B$.

By induction on $\alpha < \omega_1$ define $p_\alpha \in \mathbb{P}_{\vec{K}}$ as follows:

- $p_0 := p$.
- $p_{\alpha+1}$ is the $\Delta$-least $q \leq p_\alpha$ with $q \in D_\alpha$ and $\mathrm{dom}(q) \in B$.
- $p_\alpha := \bigcup_{\beta<\alpha} p_\beta$ if $\alpha$ is a limit ordinal.

Here note that if $\alpha$ is a limit ordinal, then $p_\alpha \in K_{\mathrm{dom}(p_\alpha)}$, and so $p_\alpha \in \mathbb{P}_{\vec{K}}$. This is because $\mathrm{dom}(p_\alpha) \in B$, and $p_\alpha$ can be recovered from $p$, $\Delta \cap \mathbb{P}_{<\mathrm{dom}(p_\alpha)}$, $\langle D_\beta \cap \mathbb{P}_{<\mathrm{dom}(p_\alpha)} \mid \beta < \alpha\rangle$ and $B \cap \mathrm{dom}(p_\alpha)$, which are all in $K_{\mathrm{dom}(\alpha)}$.

For each $\alpha < \omega_1$ let $\gamma_\alpha$ be such that $p_{\alpha+1} \Vdash$ " $\min(\dot{C} \setminus \alpha) = \gamma_\alpha$ ". Then, because $S$ is stationary, we can take a limit ordinal $\alpha^* \in S$ such that $\mathrm{dom}(p_{\alpha^*}) = \alpha^*$ and such that $\gamma_\alpha < \alpha^*$ for all $\alpha < \alpha^*$. Then $p^* := p_{\alpha^*} \leq p$. Moreover it holds that $p^* \Vdash$ " $\alpha^* \in \dot{C}$ ". Therefore these $p^*$ and $\alpha^*$ are as desired.

(2) In $V$ let $H := \mathcal{H}_{\omega_2}$ and $X := X_{\vec{K}}$. Suppose that $G$ is a $\mathbb{P}_{\vec{K}}$-generic filter over $V$. We show that $X$ is non-stationary in $[H]^\omega$ in $V[G]$.

First note that $\bigcup G : \omega_1 \to 2$ by Lem.2.6 and that $\bigcup G \notin V$. Let $F : {}^{\omega_1}2 \cap V \to \omega_1$ be the function defined as follows:

$$F(f) \ :=\ \text{the least } \gamma < \omega_1 \text{ with } f(\gamma) \neq (\textstyle\bigcup G)(\gamma).$$

We show that any $M \in X$ is not closed under $F$. Fix $M \in X$, and let $\alpha := M \cap \omega_1$. Then $K_\alpha = \mathrm{tcol}(M)$, and $(\bigcup G) \restriction \alpha \in K_\alpha$ by the construction of $\mathbb{P}_{\vec{K}}$. So there exists $f \in {}^{\omega_1}2 \cap M$ such that $\tau(f) = (\bigcup G) \restriction \alpha$, where $\tau : M \to \mathrm{tcol}(M)$ is the collapsing map. But $\tau(f) = f \restriction \alpha$, and thus $f \restriction \alpha = (\bigcup G) \restriction \alpha$. Then $F(f) \geq \alpha = M \cap \omega_1$. Therefore $M$ is not closed under $F$. $\square$

# 3 Consistency of $\diamondsuit^{++}$

In this section we discuss the consistency of $\diamondsuit^{++}$. First we prove that $\diamondsuit^{++}$ holds in $L$:

**Proposition 1.4.** *$\diamondsuit^{++}$ holds in $L$.*

*Proof.* We work in $L$. For each ordinal $\gamma$ and each $x \subseteq L_\gamma$ let $\mathrm{Sk}^\gamma(x)$ denotes the Skolem hull of $x$ in $\langle L_\gamma, \in, <_L \rangle$, that is, the smallest $M \prec \langle L_\gamma, \in, <_L \rangle$ with $x \subseteq M$.

For each $\alpha < \omega_1$ define $K_\alpha$ as follows: If there are $\beta_\alpha$, $\gamma_\alpha$, $\sigma_\alpha$ and $a_\alpha$ such that

- $\alpha < \beta_\alpha < \gamma_\alpha < \omega_1$,
- $\sigma_\alpha : L_{\gamma_\alpha} \to L_{\omega_3}$ is an elementary embedding with $\sigma_\alpha(\beta_\alpha) = \omega_2$ and $\sigma_\alpha(\alpha) = \omega_1$,
- $a_\alpha \in L_{\gamma_\alpha}$, and $L_{\gamma_\alpha} = \mathrm{Sk}^{\gamma_\alpha}(\{a_\alpha\})$.

then let $K_\alpha := L_{\beta_\alpha}$. Otherwise, take $\delta_\alpha$ such that

- $\alpha < \delta_\alpha < \omega_1$,
- $L_{\delta_\alpha} \models$ " $|\alpha| = \omega$ ",

and let $K_\alpha := L_{\delta_\alpha}$.

We show that $\vec{K} = \langle K_\alpha \mid \alpha < \omega_1 \rangle$ is a $\diamondsuit^{++}$-sequence. Clearly $\vec{K}$ satisfies the property (i) in Def.1.2. We check the properties (ii) and (iii).

First we check the property (ii). Take an arbitrary $A \subseteq \omega_1$. We show that

$$C := \{\alpha < \omega_1 \mid \mathrm{Sk}^{\omega_2}(\alpha \cup \{A\}) \cap \omega_1 = \alpha\} .$$

witnesses the property (ii).

Suppose that $\alpha \in C$. We must show that both $A \cap \alpha$ and $C \cap \alpha$ belong to $K_\alpha$. Let $M := \mathrm{Sk}^{\omega_2}(\alpha \cup \{A\})$. Moreover suppose that $\mathrm{tcol}(M) = L_\beta$, and let $\tau : M \to L_\beta$ be the collapsing map.

**Claim 1.** $L_\beta \in K_\alpha$.

*Proof of Claim 1.* If $\alpha$ is as in the latter case of the definition of $K_\alpha$, then the claim follows from the fact that $\alpha$ is countable in $L_{\delta_\alpha} = K_\alpha$ and the fact that

$\alpha = (\omega_1)^{L_\beta}$. Thus assume that $\alpha$ is as in the former case. For the contradiction assume that $\beta \geq \beta_\alpha$.

First suppose that $\beta > \beta_\alpha$. Note that $\beta_\alpha$ is not a cardinal in $L_\beta$ because $\alpha$ is the largest cardinal in $L_\beta$. Then $\beta > \gamma_\alpha$ because $\beta_\alpha$ is a cardinal in $L_{\gamma_\alpha}$. Then $\gamma_\alpha$ must be countable in $L_\beta$ because $\mathrm{Sk}^{\gamma_\alpha}(\{a_\alpha\}) = L_{\gamma_\alpha}$. So $\alpha$ is countable in $L_\beta$, too. This contradicts that $\alpha = (\omega_1)^{L_\beta}$.

Next suppose that $\beta = \beta_\alpha$. First note that $L_\beta = \mathrm{Sk}^\beta(\alpha \cup \{\tau(A)\})$. Then $|\beta_\alpha| = |\beta| = \alpha$ in $L_{\gamma_\alpha}$. This contradicts that $\beta_\alpha$ is a cardinal in $L_{\gamma_\alpha}$. $\square_{\text{Claim1}}$

From Claim 1 it follows that $A \cap \alpha = \tau(A) \in K_\alpha$. Moreover

$$C \cap \alpha \;=\; \{\alpha' < \alpha \mid \mathrm{Sk}^\beta(\alpha' \cup \{\tau(A)\}) \cap \alpha = \alpha'\} \in K_\alpha \;.$$

This completes the check of the property (ii).

Next we check the property (iii). Take an arbitrary $F : [L_{\omega_2}]^{<\omega} \to L_{\omega_2}$. It suffices to find $M \in [L_{\omega_2}]^\omega$ such that $M$ is closed under $F$, such that $M \cap \omega_1 \in \omega_1$ and such that $\mathrm{tcol}(M) = K_{M \cap \omega_1}$.

Let $N := \mathrm{Sk}^{\omega_3}(\{F\})$ and $\alpha := N \cap \omega_1 \in \omega_1$. Moreover suppose that $\mathrm{tcol}(N) = L_\gamma$, and let $\tau : N \to L_\gamma$ be the collapsing map. Then $\beta_\alpha := \tau(\omega_2)$, $\gamma_\alpha := \gamma$, $a_\alpha := \tau(F)$ and $\sigma_\alpha := \tau^{-1}$ witnesses that $\alpha$ is as in the former case of the definition of $K_\alpha$. Let $M := N \cap L_{\omega_2}$. Then $M \cap \omega_1 = \alpha \in \omega_1$, $\mathrm{tcol}(M) = L_{\beta_\alpha} = K_\alpha$, and $M$ is closed under $F$. $\square$

Next we show that $\diamondsuit^{++}$ can be forced:

**Proposition 1.5.** *There exists a poset forcing $\diamondsuit^{++}$.*

Because $\mathsf{CH}$ can be forced, it suffices to prove that under $\mathsf{CH}$ there exists a poset forcing $\diamondsuit^{++}$. We prove that the following poset forces $\diamondsuit^{++}$ under $\mathsf{CH}$:

**Definition 3.1.** *Let $\mathbb{P}(\diamondsuit^{++})$ be the poset of all $(k, \mathcal{B})$ with the following properties:*

*(i) $k$ is a function such that $\mathrm{dom}(k) < \omega_1$.*

*(ii) For each $\alpha \in \mathrm{dom}(k)$, $k(\alpha)$ is a countable transitive model of $\mathsf{ZFC}^-$ with $k \restriction \alpha \in k(\alpha)$.*

*(iii) $\mathcal{B}$ is a countable subset of ${}^{<\omega}(\omega_1)$*

*$(k_0, \mathcal{B}_0) \leq (k_1, \mathcal{B}_1)$ in $\mathbb{P}(\diamondsuit^{++})$ if the following hold:*

(i) $k_0 \supseteq k_1$, and $\mathcal{B}_0 \supseteq \mathcal{B}_1$.

(ii) $B \restriction {}^{<\omega}\alpha \in k_0(\alpha)$ for any $B \in \mathcal{B}_1$ and any $\alpha \in \mathrm{dom}(k_0) \setminus \mathrm{dom}(k_1)$.

Before proving that $\mathbb{P}(\diamondsuit^{++})$ forces $\diamondsuit^{++}$, note the following properties of $\mathbb{P}(\diamondsuit^{++})$, which can be easily proved by the standard arguments:

**Lemma 3.2.** *$\mathbb{P}(\diamondsuit^{++})$ is $\sigma$-closed and has the $< (2^\omega)^+$-c.c. Thus $\mathbb{P}(\diamondsuit^{++})$ preserves $\omega_1$. Moreover if $\mathsf{CH}$ holds in $V$, then $\mathbb{P}(\diamondsuit^{++})$ preserves all cardinals.*

**Lemma 3.3.** *Suppose that $G$ is a $\mathbb{P}(\diamondsuit^{++})$-generic filter over $V$, and let $K := \bigcup\{k \mid \exists\mathcal{B},\ (k,\mathcal{B}) \in G\}$. Then the following hold in $V[G]$:*

*(1)* $\mathrm{dom}(K) = \omega_1$.

*(2) For any $B \subseteq {}^{<\omega}(\omega_1)$ in $V$ there exists $\gamma < \omega_1$ such that $B \cap {}^{<\omega}\alpha \in K(\alpha)$ for all $\alpha \in \omega_1 \setminus \gamma$.*

*(3) $G$ is equal to the collection of all $(k,\mathcal{B}) \in \mathbb{P}(\diamondsuit^{++})$ such that $k \subseteq K$ and such that $B \restriction {}^{<\omega}\alpha \in K(\alpha)$ for all $\alpha \in \omega_1 \setminus \mathrm{dom}(k)$ and all $B \in \mathcal{B}$.*

Now we prove that $\mathbb{P}(\diamondsuit^{++})$ forces $\diamondsuit^{++}$ under $\mathsf{CH}$:

**Lemma 3.4.** *Assume $\mathsf{CH}$. Then $\mathbb{P}(\diamondsuit^{++})$ forces $\diamondsuit^{++}$.*

In the proof we use the following notation:

**Notation 3.5.** *Let $\mathbb{P}$ be a poset and $M$ be a set. $g$ is called an $(M,\mathbb{P})$-generic filter if $g$ is a filter on $\mathbb{P} \cap M$, and $g$ intersects all dense subset of $\mathbb{P}$ which is in $M$. Moreover for a $\mathbb{P}$-generic filter $G$ over $V$ let*

$$M[G] \ := \ \{\dot{a}^G \mid \dot{a} \text{ is a } \mathbb{P}\text{-name in } M\}\ ,$$

*where $\dot{a}^G$ denote the evaluation of $\dot{a}$ by $G$.*

*Proof of Lem.3.4.* Let $\mathbb{P} := \mathbb{P}(\diamondsuit^{++})$. Suppose that $G$ is a $\mathbb{P}$-generic filter over $V$ and let $K := \bigcup\{k \mid \exists\mathcal{B},\ (k,\mathcal{B}) \in G\}$. We show that $K$ $(= \langle K(\alpha) \mid \alpha < \omega_1\rangle)$ is a $\diamondsuit^{++}$-sequence in $V[G]$.

$\mathrm{dom}(K) = \omega_1$ by Lem.3.3 (1), and clearly $K$ satisfies the property (i) in the definition of $\diamondsuit^{++}$ (Def.1.2). So we check the properties (ii) and (iii).

First we check the property (ii) in Def.1.2. We work in $V[G]$. Take an arbitrary $A \subseteq \omega_1$. We construct a club $C \subseteq \omega_1$ such that both $A \cap \alpha$ and $C \cap \alpha$ belong to $K(\alpha)$ for all $\alpha \in C$.

Let $\dot{A} \in V$ be a $\mathbb{P}$-name for $A$. Because $\mathbb{P}$ has the $<\omega_2$-c.c., we may assume that $\dot{A} \in (\mathcal{H}_{\omega_2})^V$. In $V$ take a sufficiently large regular cardinal $\theta$ and an $N' \prec \langle (\mathcal{H}_\theta)^V, \in, \mathbb{P}(\diamondsuit^{++}), \dot{A} \rangle$ of size $\omega_1$ such that $N' \cap (\mathcal{H}_{\omega_2})^V$ is transitive. Let $N := \mathrm{tcol}(N')$, $\tau : N' \to N$ be the collapsing map and $\mathbb{Q} := \tau(\mathbb{P})$. Note that $\tau(\dot{A}) = \dot{A}$ because $\dot{A} \in (\mathcal{H}_{\omega_2})^V$, and $N' \cap (\mathcal{H}_{\omega_2})^V$ is transitive. Note also that $\tau[G \cap N'] = G \cap N' = G \cap N$ because $\mathbb{P}(\diamondsuit^{++}) \subseteq (\mathcal{H}_{\omega_2})^V$ and that $G \cap N$ is $\mathbb{Q}$-generic over $N$.

In $V$ take a surjection $\pi : \omega_1 \to N$ such that $\pi(0) = \mathbb{Q}$ and $\pi(1) = \dot{A}$, and let $E := \{(\beta, \alpha) \in {}^2(\omega_1) \mid \pi(\beta) \in \pi(\alpha)\}$. Note that $N = \mathrm{tcol}(\omega_1, E)$ and the collapsing map is $\pi$. Moreover for each $\alpha \in \omega_1 \setminus 2$ let $N_\alpha := \mathrm{tcol}(\alpha, E \cap {}^2\alpha)$, $\pi_\alpha : \alpha \to N_\alpha$ be the collapsing map, $\mathbb{Q}_\alpha := \pi_\alpha(0)$, and $\dot{A}_\alpha := \pi_\alpha(1)$.

In $V[G]$ take $\delta \in \omega_1 \setminus 2$ such that $E \cap {}^2\alpha \in K(\alpha)$ for all $\alpha \in \omega_1 \setminus \delta$. Then let $C$ be the set of all $\alpha \in \omega_1 \setminus \delta$ with the following property:

- $\langle \alpha, E \cap {}^2\alpha \rangle \prec \langle \omega_1, E \rangle$.
- Letting $G_\alpha$ be the set of all $(k, \mathcal{B}) \in \mathbb{Q}_\alpha$ such that
    - $k \subseteq K \restriction \alpha$,
    - $B \cap {}^{<\omega}\beta \in K(\beta)$ for all $\beta \in \alpha \setminus \mathrm{dom}(k)$ and all $B \in \mathcal{B}$,

  $G_\alpha$ is $\mathbb{Q}_\alpha$-generic over $N_\alpha$.
- The map $\iota_\alpha : N_\alpha[G_\alpha] \to N[G \cap N]$, defined as

  $$\iota_\alpha(\pi_\alpha(\beta)^{G_\alpha}) := \pi(\beta)^{G \cap N}$$

  for each $\beta < \alpha$ such that $\pi_\alpha(\beta)$ is a $\mathbb{Q}_\alpha$-name, is an elementary embedding.

It is not hard to see that $C$ is club in $\omega_1$. Here note that if $\alpha \in C$, then $N_\alpha, \pi_\alpha, \mathbb{Q}_\alpha, \dot{A}_\alpha, G_\alpha \in K(\alpha)$ because $E \cap {}^2\alpha, K \restriction \alpha \in K(\alpha)$, and $K(\alpha)$ is a transitive model of $\mathsf{ZFC}^-$. Thus $A \cap \alpha = (\dot{A}_\alpha)^{G_\alpha} \in K(\alpha)$ for all $\alpha \in C$. Finally note that if $\alpha \in C$, then $C \cap \alpha$ equals to the set of all $\beta < \alpha$ with the following properties:

- $\langle \beta, E \cap {}^2\beta \rangle \prec \langle \alpha, E \cap {}^2\alpha \rangle$,
- $G_\beta$ is a $\mathbb{Q}_\beta$-generic over $N_\beta$.

- The map $\iota_{\beta,\alpha} : N_\beta[G_\beta] \to N_\alpha[G_\alpha]$, defined as

$$\iota_{\beta,\alpha}(\pi_\beta(\gamma)^{G_\beta}) := \pi_\alpha(\gamma)^{G_\alpha}$$

  for each $\gamma < \beta$ such that $\pi_\beta(\gamma)$ is a $\mathbb{Q}_\beta$-name, is an elementary embedding.

Thus if $\alpha \in C$, then $C \cap \alpha$ is recovered from $E \restriction {}^2\alpha$ and $K \restriction \alpha$ in $K(\alpha)$. This completes the check of the property (ii) in Def.1.2.

Next we check the property (iii) in Def.1.2. We work in $V$. Let $\dot{G}$ be the canonical $\mathbb{P}$-name for a $\mathbb{P}$-generic filter. Moreover let $\dot{K}$ be the canonical $\mathbb{P}$-name for $\bigcup\{k \mid \exists\mathcal{B},\ (k,\mathcal{B}) \in \dot{G}\}$.

Take an arbitrary $(k,\mathcal{B}) \in \mathbb{P}$ and an arbitrary $\mathbb{P}$-name $\dot{F}$ for a function from $[(\mathcal{H}_{\omega_2})^{V^{\mathbb{P}}}]^{<\omega}$ to $(\mathcal{H}_{\omega_2})^{V^{\mathbb{P}}}$. It suffices to find $(k^*,\mathcal{B}^*) \leq (k,\mathcal{B})$ and a $\mathbb{P}$-name $\dot{x}$ of a countable subset of $(\mathcal{H}_{\omega_2})^{V^{\mathbb{P}}}$ such that $(k^*,\mathcal{B}^*)$ forces that $\dot{x}$ is closed under $\dot{F}$, that $\dot{x} \cap \omega_1 \in \omega_1$ and that $\mathrm{tcol}(\dot{x}) = \dot{K}(\dot{x} \cap \omega_1)$.

Take a sufficiently large regular cardinal $\theta$ and a countable elementary submodel $\bar{M}$ of $\langle \mathcal{H}_\theta, \in, \mathbb{P}, (h,\mathcal{B}), \dot{F}\rangle$. Let $M := \bar{M} \cap \mathcal{H}_{\omega_2}$ and $\alpha := \bar{M} \cap \omega_1$. Furthermore take an $(\bar{M},\mathbb{P})$-generic filter $g$ containing $(k,\mathcal{B})$, and let

$$\begin{aligned} k^{**} &:= \bigcup\{k' \mid \exists\mathcal{B}',\ (k',\mathcal{B}') \in g\}\ , \\ \mathcal{B}^* &:= \bigcup\{\mathcal{B}' \mid \exists k',\ (k',\mathcal{B}') \in g\}\ . \end{aligned}$$

Note that $\mathrm{dom}(k^{**}) = \alpha$ and that $\mathcal{B}^* = \mathcal{P}({}^{<\omega}(\omega_1)) \cap \bar{M}$. Note also that $(k^{**},\mathcal{B}^*)$ is a lower bound of $g$.

Let $\bar{K}_\alpha := \mathrm{tcol}(\bar{M})$, and $\tau : \bar{M} \to \bar{K}_\alpha$ be the transitive collapse, and let $K_\alpha := \tau[M] = \tau(\mathcal{H}_{\omega_2})$. Note that $\tau[g]$ is a $\tau(\mathbb{Q})$-generic filter over $\bar{K}_\alpha$. Let $k^*$ be the function on $\alpha+1$ which extends $k^{**}$ and such that $k^*(\alpha) = K_\alpha[\tau[g]]$.

We show that $(k^*,\mathcal{B}^*)$ and $\dot{x} := M[\dot{G}]$ are as desired. First remark that $k^* \restriction \alpha = k^{**} \in K_\alpha[\tau[g]]$. Moreover for all $B \in \mathcal{P}({}^{<\omega}(\omega_1)) \cap \bar{M} = \mathcal{B}^*$ we have that $B \cap {}^2\alpha = \tau(B) \in K[\tau[g]]$. Thus $(k^*,\mathcal{B}^*) \leq (k^{**},\mathcal{B}^*)$, In particular $(k^*,\mathcal{B}^*) \leq (k,\mathcal{B})$. Moreover $(k^*,\mathcal{B}^*)$ is a lower bound of $g$. Hence $(k^*,\mathcal{B}^*)$ forces that $M[\dot{G}] \cap \omega_1 = M \cap \omega_1 = \alpha$ and that $\mathrm{tcol}(M[\dot{G}]) = K_\alpha[\tau[g]] = \dot{K}(\alpha)$. Finally $(h^*,\mathcal{B}^*)$ forces that $M[\dot{G}]$ is closed under $\dot{F}$ by the $(\bar{M},\mathbb{P})$-genericity of $g$. Therefore $(k^*,\mathcal{B}^*)$ and $\dot{x} = M[\dot{G}]$ are as desired. $\square$

This completes the proof of Prop.1.5.