Semi-stationary reflection and weak square

Hiroshi Sakai

Results in this note will be included in a forthcoming joint paper with Fuchino, Torres and Usuba.

1 Introduction

In this note we investigate how weak square principles are denied by the semi-stationary reflection principle. First recall the semi-stationary reflection principle, introduced by Shelah [9]:

Notation 1.1. *For countable sets x and y let*

$$x \trianglelefteq y \overset{\text{def}}{\Leftrightarrow} x \subseteq y \ \wedge \ x \cap \omega_1 = y \cap \omega_1 \ .$$

Definition 1.2 (Shelah [9])**.** *Let W be a set with $\omega_1 \subseteq W$. We say that $X \subseteq [W]^\omega$ is* semi-stationary *in $[W]^\omega$ if the set $\{y \in [W]^\omega \mid \exists x \in X,\ x \trianglelefteq y\}$ is stationary in $[W]^\omega$.*

Definition 1.3 (Shelah [9])**.** *Let W be a set with $\omega_1 \subseteq W$. Then* the semi-stationary reflection principle in $[W]^\omega$*, denoted as $\mathsf{SSR}([W]^\omega)$, is the following statement:*

$$\mathsf{SSR}([W]^\omega) \ \equiv \ \text{For every semi-stationary } X \subseteq [W]^\omega \text{ there exists } W' \subseteq W \text{ such that } |W'| = \omega_1 \subseteq W' \text{ and such that } X \cap [W']^\omega \text{ is semi-stationary in } [W']^\omega.$$

Let the semi-stationary reflection principle*, denoted as SSR, be the statement that $\mathsf{SSR}([W]^\omega)$ holds for every $W \supseteq \omega_1$.*

Shellac [9] proved that SSR holds if and only if every ω_1-stationary preserving forcing notion is semi-proper. The following is also known:

Fact 1.4. *Both Martin's Maximum and Rado's Conjecture imply SSR.*

Next we turn our attention to weak square principles. In this note we discuss the following weak square principles, formulated by Schimmerling [8]:

Definition 1.5 (Schimmerling [8]). *Let λ be an uncountable cardinal and μ be a cardinal with $1 \leq \mu \leq \lambda$. Then let $\square_{\lambda,\mu}$ be the following statement:*

$\square_{\lambda,\mu}$ $\equiv$ *There exists a sequence $\langle \mathcal{C}_\alpha \mid \alpha < \lambda^+ \rangle$ such that*

(i) $\mathcal{C}_\alpha$ is a set consisting of club subsets of α,

(ii) if $\mathrm{cf}(\alpha) < \lambda$, then each $c \in \mathcal{C}_\alpha$ has the order-type $< \lambda$,

(iii) $1 \leq |\mathcal{C}_\alpha| \leq \mu$,

(iv) if $c \in \mathcal{C}_\alpha$, and $\beta \in \mathrm{Lim}(c)$, then $c \cap \beta \in \mathcal{C}_\alpha$.

Let $\square_{\lambda,<\mu}$ be the statement obtained by replacing the property (iii) in $\square_{\lambda,\mu}$ with

(v) $1 \leq |\mathcal{C}_\alpha| < \mu$.

A sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \lambda^+ \rangle$ witnessing $\square_{\lambda,\mu}$ or $\square_{\lambda,<\mu}$ is called a $\square_{\lambda,\mu}$-sequence or a $\square_{\lambda,<\mu}$-sequence, respectively.

Many large cardinal properties are known to imply the failure of the above weak square principles. Among other things, sharp results have been obtained on for what λ and μ Martin's Maximum denies $\square_{\lambda,\mu}$ (See Devlin [1] for the proof of Fact 1.6 (2)):

Fact 1.6 ((1),(3),(4): Cumming-Magidor [3], (2): Baumgartner). *Assume Martin's Maximum. Then we have the following:*

(1) $\square_{\lambda,<\mathrm{cf}(\lambda)}$ fails for any uncountable cardinal λ.

(2) $\square_{\omega_1,\omega_1}$ fails.

(3) $\square_{\lambda,\lambda}$ fails for any singular cardinal λ of cofinality ω.

(4) $\square_{\lambda,<\lambda}$ fails for any singular cardinal λ of cofinality ω_1.

Fact 1.7 (Cummings-Magidor [3]). *If there exists a supercompact cardinal, then there exists a class forcing extension in which Martin's Maximum together with both (I) and (II) below holds:*

(I) $\square_{\lambda,\mathrm{cf}(\lambda)}$ holds for every cardinal λ with $\mathrm{cf}(\lambda) > \omega_1$.

(II) $\square_{\lambda,\lambda}$ holds for every singular cardinal λ of cofinality ω_1.

As for Rado's Conjecture Todorčević-Torres [12] obtained the following:

Fact 1.8 (Friedman-Krueger [5], Todorčević-Torres [12]). *Assume Rado's Conjecture. Then we have the following:*

(1) $\square_{\lambda,<\mathrm{cf}(\lambda)}$ *fails for any uncountable cardinal* λ*.*

(2) If CH *fails in addition, then* $\square_{\omega_1,\omega_1}$ *fails.*

(3) $\square_{\lambda,\lambda}$ *fails for any singular cardinal* λ *of cofinality* ω*.*

Moreover by the same argument as Fact 1.7 we can prove the following:

Fact 1.9. *If there exists a supercompact cardinal, then there exists a class forcing extension in which Rado's Conjecture together with both (I) and (II) below holds:*

(I) $\square_{\lambda,\mathrm{cf}(\lambda)}$ *holds for all cardinals* λ *with* $\mathrm{cf}(\lambda) > \omega_1$*.*

(II) $\square_{\lambda,\lambda}$ *holds for all singular cardinals* λ *of cofinality* ω_1*.*

Here note that (2) of Fact 1.8 is optimal because Rado's Conjecture is consistent with CH, and CH implies $\square_{\omega_1,\omega_1}$. So Fact 1.8 is optimal except for the case when λ is a singular cardinal of cofinality ω_1. Todorčević-Torres [12] asked the following question:

Question 1.10. *How weak square principles at singular cardinals of cofinality* ω_1 *does Rado's Conjecture deny?*

In this note we will prove the following:

Main Theorem. *Assume* SSR*. Then* $\square_{\lambda,<\lambda}$ *fails for any singular cardinal* λ *of cofinality* ω_1*.*

In particular, by Fact 1.4 we will obtain an answer to Question 1.10:

Corollary 1.11. *Assume Rado's Conjecture. Then* $\square_{\lambda,<\lambda}$ *fails for any singular cardinal* λ *of cofinality* ω_1*.*

Note that this is also optimal by Fact 1.9.

Note also that (1), (2) and (3) of Fact 1.8 also follow from SSR:

(1) Cummings-Magidor [3] proved that for any uncountable cardinal λ if every stationary subset of $E^{\lambda^+}_\omega$ reflects, then $\square_{\lambda,<\mathrm{cf}(\lambda)}$ fails. Moreover it was proved by Sakai [6] that for any regular cardinal $\mu \geq \omega_2$ if $\mathsf{SSR}([\mu]^\omega)$ holds, then every stationary subset of E^μ_ω reflects.

(2) It is known, due to Foreman-Magidor-Shelah [4], that SSR implies the Strong Chang's Conjecture. Next, Todorčević [11] proved that the Strong Chang's Conjecture implies the stationary reflection in $[\omega_2]^\omega$. Finally it is known, due to Friedman-Krueger [5], that if the stationary reflection in $[\omega_2]^\omega$ holds, and CH fails, then $\square_{\omega_1,\omega_1}$ fails.

(3) Let λ be a singular cardinal of cofinality ω. Sakai-Veličković [7] proved that if $\mathsf{SSR}([\lambda^+]^\omega)$ holds, then for any sequence $\langle \lambda_n \mid n < \omega \rangle$ of regular cardinals converging to λ there are no better scales of length λ^+ in $\Pi_{n<\omega}\lambda_n$. Moreover it was proved in Cummings-Foreman-Magidor [2] that the latter statement (the non-existence of better scales) implies the failure of $\square_{\lambda,\lambda}$.

Thus we have the following:

Corollary 1.12. *Assume* SSR*. Then we have the following:*

(1) $\square_{\lambda,<\mathrm{cf}(\lambda)}$ *fails for any uncountable cardinal* λ*.*

(2) If CH *fails in addition, then* $\square_{\omega_1,\omega_1}$ *fails.*

(3) $\square_{\lambda,\lambda}$ *fails for any singular cardinal* λ *of cofinality* ω*.*

(4) $\square_{\lambda,<\lambda}$ *fails for any singular cardinal* λ *of cofinality* ω_1*.*

Of course this is optimal.

2 Preliminaries

Here we give notation and basic facts used in this note.

First we present miscellaneous notations:

Let A be a set of ordinals. $\mathrm{Lim}(A)$ denotes the set of all limit points in A, i.e. $\mathrm{Lim}(A) = \{\alpha \in A \mid \sup(A \cap \alpha) = \alpha\}$. Let $\bar{\sup}(A) := \sup\{\alpha + 1 \mid \alpha \in A\}$, i.e. $\bar{\sup}(A)$ is the least $\beta \in \mathrm{On}$ such that $\beta > \alpha$ for all $\alpha \in A$. Note that if A does not have the greatest element, then $\bar{\sup}(A) = \sup(A)$. If $A = \{a_i \mid i \in I\}$ for some index set I, then $\bar{\sup}(A)$ is also denoted as $\bar{\sup}_{i\in I} a_i$.

For a regular cardinal μ and a limit ordinal $\lambda > \mu$ we let E^λ_μ denote the set $\{\alpha < \lambda \mid \mathrm{cf}(\alpha) = \mu\}$. For an ordinal η let $\eta \bmod \omega$ be $n \in \omega$ such that $\eta - n$ is a limit ordinal. Let A be a set and $F : {}^{<\omega}A \to A$ be a function. Then for each $B \subseteq A$ let $\mathrm{cl}_F(B)$ denote the closure of B under F, i.e. $\mathrm{cl}_F(B)$ is the smallest $\bar{B} \supseteq B$ such that $F[{}^{<\omega}\bar{B}] \subseteq \bar{B}$.

Next let $\mathcal{M}$ be a structure of a countable language in which a well-ordering of its universe can be defined. For $A \subseteq \mathcal{M}$ let $\mathrm{Sk}^{\mathcal{M}}(A)$ denote the Skolem hull of A in $\mathcal{M}$, i.e. $\mathrm{Sk}^{\mathcal{M}}(A)$ is the smallest $M \prec \mathcal{M}$ with $A \subseteq M$. We use the following fact:

Fact 2.1 (folklore)**.** *Let* θ *be a regular uncountable cardinal,* Δ *be a well-ordering of* $\mathcal{H}_\theta$ *and* $\mathcal{M}$ *be a structure obtained by adding countable many constants, functions and predicates to* $\langle \mathcal{H}_\theta, \in, \Delta\rangle$*. Suppose that* $B \subseteq A \in M \prec \mathcal{M}$*. Let* $\bar{M} := \mathrm{Sk}^{\mathcal{M}}(M \cup B)$*. Then the following hold:*

(1) $\bar{M} = \{f(b) \mid f : {}^{<\omega}A \to \mathcal{H}_\theta,\ f \in M,\ b \in {}^{<\omega}B\}$*.*

(2) $\bar{\sup}(\bar{M} \cap \lambda) = \bar{\sup}(M \cap \lambda)$ *for any regular cardinal* $\lambda \in M$ *with* $|A| < \lambda$*.*

Proof. (1) Let N be the set in the right side of the equation. First note that $M \cup B \subseteq N$ and that if $M \cup B \subseteq N' \prec \mathcal{M}$, then $N \subseteq N'$. So it suffices to show that $N \prec \mathcal{M}$. We use the Tarski-Vaught criterion.

Suppose that φ is a formula, that $d \in {}^{<\omega}N$ and that $\mathcal{M} \models \exists v\varphi[v, d]$. It suffices to find $c \in N$ such that $\mathcal{M} \models \varphi[c, d]$.

Because $d \in {}^{<\omega}N$, we can take a function $f : {}^{<\omega}A \to \mathcal{H}_\theta$ in M and $b \in {}^{<\omega}B$ such that $d = f(b)$. Then there exists a function $g : {}^{<\omega}A \to \mathcal{H}_\theta$ such that for any $a \in {}^{<\omega}A$ if $\mathcal{M} \models \exists v\varphi[v, f(a)]$, then $\mathcal{M} \models \varphi[g(a), f(a)]$. We can take such g in M by the elementarity of M.

Then $c := g(b) \in N$. Moreover $\mathcal{M} \models \varphi[c, d]$ by the choice of g and the assumption that $\mathcal{M} \models \exists v\varphi[v, d]$. Therefore c is as desired.

(2) Fix a regular cardinal $\lambda \in M$ with $|A| < \lambda$. Clearly $\sup(\bar{M} \cap \lambda) \geq \sup(M \cap \lambda)$. For the converse inequality note that for any $f : {}^{<\omega}A \to \mathcal{H}_\theta$ in M and any $b \in {}^{<\omega}B$ it holds that

$$f(b) \;\leq\; \sup\{f(a) \mid a \in {}^{<\omega}A \wedge f(a) \in \lambda\} \;\in\; M \cap \lambda \,.$$

by the elementarity of M and the fact that $\mathrm{cf}(\lambda) = \lambda > |A|$. Then by (1) we have that $\sup(\bar{M} \cap \lambda) \leq \sup(M \cap \lambda)$. □

Next we give notation and a fact on PCF theory due to Shelah:

Let λ be a singular cardinal and $\langle \lambda_\xi \mid \xi < \mathrm{cf}(\lambda) \rangle$ be an increasing sequence of regular cardinals converging to λ. For $h, h' \in \Pi_{\xi < \mathrm{cf}(\lambda)} \lambda_\xi$ and $\zeta < \mathrm{cf}(\lambda)$ let

$$h <_\zeta h' \;\overset{\text{def}}{\Leftrightarrow}\; \forall \xi \in \mathrm{cf}(\lambda) \setminus \zeta,\; h(\xi) < h'(\xi) \,.$$

Then let

$$h <^* h' \;\overset{\text{def}}{\Leftrightarrow}\; \exists \zeta < \mathrm{cf}(\lambda),\; h <_\zeta h' \,.$$

Also let "$\leq_\zeta$" and "$\leq^*$" be the orders obtained by replacing "$<$" in the definition of "$<_\zeta$" and "$<^*$" with "$\leq$".

A $<^*$-increasing $<^*$-cofinal sequence in $\Pi_{\xi < \mathrm{cf}(\lambda)} \lambda_\xi$ of a regular length is called a *scale in* $\Pi_{\xi < \mathrm{cf}(\lambda)} \lambda_\xi$. The following is well-known:

Fact 2.2 (Shelah [10])**.** *For every singular cardinal λ there exists a strictly increasing sequence $\langle \lambda_\xi \mid \xi < \mathrm{cf}(\lambda) \rangle$ of regular cardinals converging to λ such that a scale in $\Pi_{\xi < \mathrm{cf}(\lambda)} \lambda_\xi$ of length λ^+ exists.*

Next we give our notation on trees. In this note we only deal with subtrees of ${}^{<\omega}\mathrm{On}$. For some technical reasons we use not only trees of height ω but also those of finite height.

$T \subseteq {}^{<\omega}\mathrm{On}$ is called a *tree* if T is closed under initial segments. Let $T \subseteq {}^{<\omega}\mathrm{On}$ be a tree. The height of T, denoted as $\mathrm{ht}(T)$, is $\bar{\sup}\{\mathrm{dom}(t) \mid t \in T\}$. For $A \subseteq \omega$ we let T_A denote the set $\{t \in T \mid \mathrm{dom}(t) \in A\}$. So $T_{\{n\}}$ is the n-th level of T for $n < \omega$. Let $T_{\mathrm{even}} := T_{\{n<\omega \mid n:\,\mathrm{even}\}}$ and $T_{\mathrm{odd}} := T_{\{n<\omega \mid n:\,\mathrm{odd}\}}$. For each $t \in T$ let $\mathrm{Suc}_T(t)$ denote the set $\{\alpha \mid t^\frown\langle\alpha\rangle \in T\}$.

Suppose that $k \leq \omega$, and let $\vec{\mu} = \langle \mu_n \mid n < k \rangle$ be a sequence of regular cardinals. An *unbounded (stationary) $\vec{\mu}$-tree* is a tree $T \subseteq {}^{<\omega}\mathrm{On}$ such that

- $\mathrm{ht}(T) = \omega$ if $k = \omega$, and $\mathrm{ht}(T) = k + 1$ if $k < \omega$,
- $\mathrm{Suc}_T(t)$ is an unbounded (stationary) subset of $\mu_{\mathrm{dom}(t)}$ for each $t \in T$ with $\mathrm{dom}(t) < k$.

If $\mu_n = \nu$ for all $n < k$, then "$\vec{\mu}$-tree" is denoted as "ν-tree".

Finally we present two facts on SSR. The first one is on the cardinal arithmetic under SSR. Recall that SSR implies $2^\omega \leq \omega_2$ and SCH. The former is due to Todorčević [11], and the latter is due to Sakai-Veličković [7]. Then we have the following:

Fact 2.3. SSR *implies that $\mu^\omega = \mu$ for every regular cardinal $\mu \geq \omega_2$.*

The second one is a technical fact proved in Sakai-Veličković [7] (Lemma 2.2):

Notation 2.4. *For countable sets $x, y \subseteq \mathrm{On}$ we let $x \trianglelefteq^* y$ denote that the following (i)–(iii) hold:*

(i) $x \trianglelefteq y$.

(ii) $\bar{\sup}(x) = \bar{\sup}(y)$.

(iii) $\bar{\sup}(x \cap \alpha) = \bar{\sup}(y \cap \alpha)$ *for all $\alpha \in x$ of cofinality ω_1.*

Fact 2.5 (Sakai-Veličković [7])**.** *Let ν be a regular cardinal $\geq \omega_2$, and assume* $\mathsf{SSR}([\nu]^\omega)$. *Then for any stationary $X \subseteq [\nu]^\omega$ there exists $W \subseteq \nu$ such that $|W| = \omega_1 \subseteq W$ and such that the set $\{y \in [W]^\omega \mid \exists x \in X,\ x \trianglelefteq^* y\}$ is stationary in $[W]^\omega$.*

3 Proof of Main Theorem

Here we prove Main Theorem:

Main Theorem. *Assume* SSR*. Then $\square_{\lambda,<\lambda}$ fails for any singular cardinal λ of cofinality ω_1.*

The proof of Main Theorem is based on the proof in Sakai-Veličković [7] of the fact that SSR implies SCH. First we prove that $\square_{\lambda,<\lambda}$ implies the existence of a variant of a very good scale:

Lemma 3.1. *Let λ be a singular cardinal of cofinality ω_1, and suppose that $\square_{\lambda,<\lambda}$ holds. Then there exist an increasing sequence $\vec{\lambda} = \langle \lambda_\xi \mid \xi < \omega_1 \rangle$ of regular cardinals converging to λ, a scale $\vec{h} = \langle h_\alpha \mid \alpha < \lambda^+ \rangle$ in $\Pi_{\xi<\omega_1} \lambda_\xi$ and a stationary $S \subseteq E^{\lambda^+}_\omega$ with the following property:*

(∗) *For every* $\delta \in E^{\lambda^+}_{\omega_1}$ *there exist a club* $c \subseteq \delta$ *and* $\zeta < \omega_1$ *such that* $\langle h_\alpha \mid \alpha \in c \cap S \rangle$ *is* $<_\zeta$*-increasing.*

Proof. Let $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \lambda^+ \rangle$ be a sequence witnessing $\square_{\lambda,<\lambda}$. Then we can take a cardinal $\mu < \lambda$ and a stationary $S \subseteq E^{\lambda^+}_\omega$ such that $1 \leq |\mathcal{C}_\alpha| \leq \mu$ for all $\alpha \in S$. By Fact 2.2 take an increasing sequence $\vec{\lambda} = \langle \lambda_\xi \mid \xi < \omega_1 \rangle$ of regular cardinals converging to λ and a scale $\vec{h}' = \langle h'_\alpha \mid \alpha < \lambda^+ \rangle$ in $\Pi_{\xi<\omega_1}\lambda_\xi$.

By induction on $\alpha < \lambda^+$ define $h_\alpha \in \Pi_{\xi<\omega_1}\lambda_\xi$ as follows: If $\alpha \notin S$, then let h_α be an arbitrary element of $\Pi_{\xi<\omega_1}\lambda_\xi$ such that $h'_\alpha \leq^* h_\alpha$ and such that $h_\beta <^* h_\alpha$ for all $\beta < \alpha$. Suppose that $\alpha \in S$. First for each $c \in \mathcal{C}_\alpha$ let $h_c \in \Pi_{\xi<\omega_1}\lambda_\xi$ be such that

$$h_c(\xi) \;=\; \begin{cases} \bar{\sup}\{h_\beta(\xi) \mid \beta \in c\} & \cdots \quad \text{if } \lambda_\xi > \text{o.t.}(c), \\ 0 & \cdots \quad \text{otherwise.} \end{cases}$$

Then let $h_\alpha \in \Pi_{\xi<\omega_1}\lambda_\xi$ be such that

$$h_\alpha(\xi) \;=\; \begin{cases} \max(\, h'_\alpha(\xi)\,,\, \bar{\sup}\{h_c(\xi) \mid c \in \mathcal{C}_\alpha\}\,) & \cdots \quad \text{if } \lambda_\xi > \mu, \\ 0 & \cdots \quad \text{otherwise.} \end{cases}$$

Clearly $\vec{h} = \langle h_\alpha \mid \alpha < \lambda^+ \rangle$ is a scale in $\Pi_{\xi<\omega_1}\lambda_\xi$. To check (∗) suppose that $\alpha \in E^{\lambda^+}_{\omega_1}$. Take an arbitrary $c \in \mathcal{C}_\alpha$. Note that $\text{o.t.}(c) < \lambda$. Hence we can take $\zeta < \omega_1$ with $\text{o.t.}(c), \mu < \lambda_\zeta$. Then $\langle h_\alpha \mid \alpha \in \text{Lim}(c) \cap S \rangle$ is $<_\zeta$-increasing by the construction of $\vec{h}$ and the coherency of $\vec{\mathcal{C}}$. □

By Lem.3.1 and Fact 2.3 it suffices for Main Theorem to show the following:

Lemma 3.2. *Let* λ *be a singular cardinal of cofinality* ω_1 *such that* $\mu^\omega < \lambda$ *for all* $\mu < \lambda$*. Suppose that there exist* $\vec{\lambda}$*,* $\vec{h}$ *and* S *satisfying as in Lem.3.1. Then* $\mathsf{SSR}([\lambda^+]^\omega)$ *fails.*

Below fix λ, $\vec{\lambda}$, $\vec{h}$ and S as follows:

- λ is a singular cardinal of cofinality ω_1 such that $\mu^\omega < \lambda$ for all $\mu < \lambda$.
- $\vec{\lambda} = \langle \lambda_\xi \mid \xi < \omega_1 \rangle$ is an increasing sequence of regular cardinals converging to λ.
- $\vec{h} = \langle h_\alpha \mid \alpha < \lambda^+ \rangle$ is a scale in $\Pi_{\xi<\omega_1}\lambda_\xi$.
- S is a stationary subset of $E^{\lambda^+}_\omega$.
- $\vec{\lambda}$, $\vec{h}$ and S satisfies (∗).

Because $\mu^\omega < \lambda$ for all $\mu < \lambda$, by shrinking $\vec{\lambda}$ if necessary, we may assume the following:

- $\omega_2 < \lambda_0$, and $\lambda_\xi > (\sup_{\eta<\xi} \lambda_\eta)^\omega$ for all $\xi < \omega_1$.

Next we give a stationary $X \subseteq [\lambda^+]^\omega$ witnessing the failure of $\mathsf{SSR}([\lambda^+]^\omega)$ in the sense of Fact 2.5. For this take $\vec{b}$ and $\vec{S}$ as in the previous section:

- $\vec{b} = \langle b_\xi \mid \xi < \omega_1 \rangle$ is a pairwise almost disjoint sequence of elements of $[\omega]^\omega$.
- $\vec{S} = \langle S_0^\xi, S_1^\xi \mid \xi < \omega_1 \rangle$ is a sequence such that S_0^ξ and S_1^ξ are disjoint stationary subsets of $E_{\omega_1}^{\lambda_\xi}$.

Let X be the following set:

X := the set of all $x \in [\lambda^+]^\omega$ such that

(i) $x \cap \omega_1 \in \omega_1$.

(ii) $\bar{\sup}(x) \in S$.

(iii) $\bar{\sup}(x \cap \lambda_\eta) > h_{\bar{\sup}(x)}(\eta)$ for all $\eta \in x \cap \omega_1$.

(iv) There is $\zeta < x \cap \omega_1$ such that for every $\eta \in x \cap \omega_1 \setminus \zeta$

- $\min(x \setminus h_{\bar{\sup}(x)}(\eta)) \in S_1^\eta$ if $\eta \bmod \omega \in b_{x\cap\omega_1}$,
- $\min(x \setminus h_{\bar{\sup}(x)}(\eta)) \in S_0^\eta$ if $\eta \bmod \omega \notin b_{x\cap\omega_1}$.

All we have to prove are the following:

Lemma 3.3. *X is stationary in $[\lambda^+]^\omega$.*

Lemma 3.4. *There are no $W \subseteq \lambda^+$ such that $|W| = \omega_1 \subseteq W$ and such that the set $\{y \in [W]^\omega \mid \exists x \in X,\ x \trianglelefteq^* y\}$ is stationary.*

First we prove Lem.3.4:

Proof of Lem.3.4. For the contradiction assume that there exist such W. Note that if $x \in X$, and $x \trianglelefteq^* y$, then $y \in X$. Thus $X \cap [W]^\omega$ is stationary in $[W]^\omega$.

Let $\delta := \bar{\sup}(W)$. By Fodor's lemma take $\zeta < \omega_1$ such that

$$Y := \{x \in X \cap [W]^\omega \mid \zeta \text{ witnesses that } x \text{ satisfies (iv) of elements of } X\}$$

is stationary in $[W]^\omega$. We may assume that ζ is a limit ordinal.

First suppose that $\mathrm{cf}(\delta) < \omega_1$. Let $\alpha_n := \min(W \setminus h_\delta(\zeta + n))$ for each $n \in \omega$. Then there are club many $y \in [W]^\omega$ such that

- $\bar{\sup}(x) = \delta$,
- $\{\alpha_n \mid n < \omega\} \subseteq x$.

Hence we can take such $x_0, x_1 \in Y$ with $x_0 \cap \omega_1 \neq x_1 \cap \omega_1$ and $x_0 \cap \omega_1, x_1 \cap \omega_1 \geq \zeta + \omega$. For each $i = 0, 1$ note that $\min(x_i \setminus h_{\bar{\sup}(x_i)}(\zeta + n)) = \alpha_n$ for all $n < \omega$ because $\alpha_n \in x_i \subseteq W$, and $\bar{\sup}(x_i) = \delta$. Hence, by the property (iv) of elements of X, for each $n < \omega$, $\alpha_n \in S_1^{\zeta+n}$ if $n \in b_{x_i \cap \omega_1}$, and $\alpha_n \in S_0^{\zeta+n}$ if $n \notin b_{x_i \cap \omega_1}$. This contradicts that $S_1^{\zeta+n}$ and $S_0^{\zeta+n}$ are disjoint and that $b_{x_0 \cap \omega_1}$ and $b_{x_1 \cap \omega_1}$ are almost disjoint.

Next suppose that $\mathrm{cf}(\bar{\sup}(W)) = \omega_1$. By $(*)$ we can take a club $c \subseteq \delta$ and $\eta < \omega_1$ such that $\langle h_\alpha \mid \alpha \in c \cap S \rangle$ is $<_\eta$-increasing. We may assume that η is a limit ordinal $\geq \zeta$. For each $n < \omega$ let

$$\begin{aligned} \beta_n &:= \bar{\sup}\{h_\alpha(\eta + n) \mid \alpha \in c \cap S\} , \\ \gamma_n &:= \min(W \setminus \beta_n) . \end{aligned}$$

Then there are club many $x \in [W]^\omega$ such that

- $\bar{\sup}(x) \in c$,
- $x \cap \beta_n \subseteq \bar{\sup}\{h_\alpha(\eta + n) \mid \alpha \in c \cap S \cap \bar{\sup}(x)\}$ for all $n < \omega$, (thus if $\bar{\sup}(x) \in c \cap S$, then $x \cap \beta_n \subseteq h_{\bar{\sup}(x)}(\eta + n)$ for all $n < \omega$,)
- $\{\gamma_n \mid n < \omega\} \subseteq x$.

Hence we can take such $x_0, x_1 \in Y$ with $x_0 \cap \omega_1 \neq x_1 \cap \omega_1$ and $x_0 \cap \omega_1, x_1 \cap \omega_1 \geq \eta + \omega$. Here note that $\bar{\sup}(x_i) \in S$ for each $i = 0, 1$ by the property (ii) of elements of X. Hence $\min(x_i \setminus h_{\bar{\sup}(x_i)}(\eta + n)) = \gamma_n$ for all $n < \omega$. Then, as in the previous case, we get a contradiction by the property (iv) of elements of X. $\square$

The rest of this section is devoted to the proof of Lem.3.3.
First we prepare notation:

- For each $\xi \in \omega_1 \setminus \omega$ let $\vec{\eta}^\xi = \langle \eta_n^\xi \mid n < \omega\text{: even} \rangle$ be a one to one enumeration of ξ, and let $\vec{\eta} := \langle \vec{\eta}^\xi \mid \xi \in \omega_1 \setminus \omega \rangle$.
- For each $\xi \in \omega_1 \setminus \omega$ and each $n < \omega$ let

$$\mu_n^\xi := \begin{cases} \lambda_{\eta_n^\xi} & \cdots \text{ if } n \text{ is even,} \\ \lambda^+ & \cdots \text{ if } n \text{ is odd,} \end{cases}$$

  and let $\vec{\mu}^\xi := \langle \mu_n^\xi \mid n < \omega \rangle$ for each $\xi \in \omega_1 \setminus \omega$.
- For a set x with $|x| \leq \omega_1$ let χ_x be the function in $\Pi_{\xi < \omega_1} \lambda_\xi$ such that $\chi_x(\xi) = \bar{\sup}(x \cap \lambda_\xi)$.

Lem.3.3 will follow from Lem.3.5 and 3.6 below. Lem.3.5 will be proved later:

Lemma 3.5. *For any function $F : {}^{<\omega}(\lambda^+) \to \lambda^+$ and any $\zeta \in \omega_1 \setminus \omega$ there exist $\xi \in \omega_1 \setminus \zeta$, an unbounded $\vec{\mu}^\xi$-tree T and a function $f \in \Pi_{n:\text{even}}\mu^\xi_n$ with the following properties:*

(i) $\mathrm{cl}_F(\xi \cup \mathrm{ran}(t)) \cap \omega_1 = \xi$ *for every* $t \in T$.

(ii) For any $t \in T$ *and any even* $n \in \mathrm{dom}(t)$,

- $t(n) \in S_1^{\eta^\xi_n}$ *if* $\eta^\xi_n \bmod \omega \in b_\xi$,
- $t(n) \in S_0^{\eta^\xi_n}$ *if* $\eta^\xi_n \bmod \omega \notin b_\xi$.

(iii) $\mathrm{cl}_F(\xi \cup \mathrm{ran}(t)) \cap t(n) \subseteq f(n)$ *for all* $t \in T$ *and all even* $n \in \mathrm{dom}(t)$.

Lemma 3.6. *There are stationary many $x \in [\lambda^+]^{\omega_1}$ such that $\omega_1 \subseteq x$, such that $\bar{\sup}(x) \in S$ and such that $\chi_x \leq^* h_{\bar{\sup}(x)}$.*

Proof of Lem.3.6. Take an arbitrary function $F : {}^{<\omega}(\lambda^+) \to \lambda$. We find $x \in [\lambda^+]^{\omega_1}$ such that $\omega_1 \subseteq x$ and such that $\chi_x \leq^* h_{\bar{\sup}(x)}$.

Let θ be a suffciently large regular cardinal, and let $\mathcal{M} := \langle \mathcal{H}_\theta, \in, \vec{\lambda}, \vec{h}, F \rangle$. Then, because S is a stationary subset of $E^{\lambda^+}_\omega$, we can take an $\subseteq$-increasing $\in$-chain $\langle M_n \mid n < \omega \rangle$ in $[\mathcal{H}_\theta]^{\omega_1}$ such that $\omega_1 \subseteq M_n \prec \mathcal{M}$ for each $n < \omega$ and such that $\delta := \bar{\sup}_{n<\omega}(\bar{\sup}(M_n \cap \lambda^+)) \in S$. Let $M := \bigcup_{n<\omega} M_n$. We show that $x := M \cap \lambda^+$ is as desired. Note that $\bar{\sup}(x) = \delta$. Hence it suffices to prove that $\chi_x \leq^* h_\delta$.

For each $n < \omega$ we can take $\delta_n < \delta$ such that $\chi_{M_n} <^* h_{\delta_n}$ because $M_n \in M \prec \mathcal{M}$. Take $\zeta_n < \omega_1$ such that $\chi_{M_n} <_{\zeta_n} h_{\delta_n} <_{\zeta_n} h_\delta$ for each $n < \omega$, and let $\zeta := \bar{\sup}_{n<\omega} \zeta_n < \omega_1$. Then for each $\xi \in \omega_1 \setminus \zeta$ it holds that

$$\bar{\sup}(x \cap \lambda_\xi) \;=\; \bar{\sup}_{n<\omega}(\bar{\sup}(M_n \cap \lambda_\xi)) \;\leq\; \bar{\sup}_{n<\omega} h_{\delta_n}(\xi) \;\leq\; h_\delta(\xi)\;.$$

Therefore $\chi_x \leq_\zeta h_\delta$. □

Before proving Lem.3.5, we prove Lem.3.3 using Lem.3.5 and 3.6:

Proof of Lem.3.3. Take an arbitrary function $F : {}^{<\omega}(\lambda^+) \to \lambda^+$. We find $x \in X$ closed under F.

Let θ be a sufficiently large regular cardinal, Δ be a well-ordering of $\mathcal{H}_\theta$ and $\mathcal{M}$ be the structure $\langle \mathcal{H}_\theta, \in, \Delta, \vec{\lambda}, \vec{h}, S, \vec{S}, \vec{b}, \vec{\eta}, F \rangle$. By Lem.3.6 take $M \prec \mathcal{M}$ such that $|M| = \omega_1 \subseteq M$, such that $\delta := \bar{\sup}(M \cap \lambda^+) \in S$ and such that $\chi_M \leq^* h_\delta$. Let $\zeta < \omega_1$ be such that $\chi_M \leq_\zeta h_\delta$.

By Lem.3.5 take $\xi \in \omega_1 \setminus \zeta + 1$, an unbounded $\vec{\mu}^\xi$-tree T and $f \in \Pi_{n:\text{even}}\mu^\xi_n$ with the properties (i)–(iii) in Lem.3.5. We can take ξ, T and f in M by the elementarity of M. Here note that $f(n) \in M \cap \lambda_{\eta^\xi_n}$ for each even n.

Take an increasing sequence $\langle \delta_n \mid n < \omega \rangle$ converging to δ. Moreover let $\bar{M} := \mathrm{Sk}^{\mathcal{M}}(M \cup \lambda)$. Note that $\mathrm{s\bar{u}p}(\bar{M} \cap \lambda^+) = \delta$ by Fact 2.1. Then by induction on n we can take $\alpha_n \in \bar{M}$ so that

- $\alpha_n \in \mathrm{Suc}_T(\langle \alpha_m \mid m < n \rangle)$,
- if n is odd, then $\delta_n < \alpha_n$,
- if n is even, then $h_\delta(\eta^\xi_n) < \alpha_n$.

Let $x := \mathrm{cl}_F(\xi \cup \{\alpha_n \mid n < \omega\})$. It suffices to show that $x \in X$. Note that $x \cap \omega_1 = \xi$. In particular, x satisfies the property (i) of elements of X.

Next note that $x \subseteq \bar{M} \cap \lambda^+$ because $\xi \cup \{\alpha_n \mid n < \omega\} \subseteq \bar{M} \prec \mathcal{M}$. Hence $\delta = \sup_{n\in\omega} \delta_n \le \mathrm{s\bar{u}p}(x) \le \mathrm{s\bar{u}p}(\bar{M} \cap \lambda^+) = \delta$. So $\mathrm{s\bar{u}p}(x) = \delta$. Then it follows that x satisfies the properties (ii) and (iii) of elements of X.

It remains to check the property (iv) of elements of X. For this first note that $x \cap \alpha_n \subseteq f(n)$ for each even $n < \omega$ by the property (iii) of Lem.3.5. Hence if n is even, and $\eta^\xi_n > \zeta$, then

$$x \cap \alpha_n \;\subseteq\; f(n) \;<\; \mathrm{s\bar{u}p}(M \cap \lambda_{\eta^\xi_n}) \;\le\; h_\delta(\eta^\xi_n) \;<\; \alpha_n \;,$$

and so $\min(x \setminus h_\delta(\eta^\xi_n)) = \alpha_n$. Then the property (iv) of elements of X follows from the property (ii) of Lem.3.5. □

We must prove Lem.3.5. First we prove a weak version of Lem.3.5:

Lemma 3.7. *For any function $F : {}^{<\omega}(\lambda^+) \to \lambda^+$ and any $\zeta \in \omega_1 \setminus \omega$ there exist $\xi \in \omega_1 \setminus \zeta$, a stationary $\vec{\mu}^\xi$-tree T and a function f on T_{even} with the following properties:*

(i) $\mathrm{cl}_F(\xi \cup \mathrm{ran}(t)) \cap \omega_1 = \xi$ *for every* $t \in T$.

(ii) For any $t \in T$ *and any even* $n \in \mathrm{dom}(t)$,

- $t(n) \in S_1^{\eta^\xi_n}$ *if* $\eta^\xi_n \bmod \omega \in b_\xi$,
- $t(n) \in S_0^{\eta^\xi_n}$ *if* $\eta^\xi_n \bmod \omega \notin b_\xi$.

(iii) $f(t) < \mu^\xi_{\mathrm{dom}(t)}$ *for each* $t \in T_{\mathrm{even}}$.

(iv) $\mathrm{cl}_F(\xi \cup \mathrm{ran}(t)) \cap t(n) \subseteq f(t \restriction n)$ *for all* $t \in T$ *and all even* $n \in \mathrm{dom}(t)$.

Proof. Take an arbitrary function $F : {}^{<\omega}(\lambda^+) \to \lambda^+$ and an arbitrary $\zeta \in \omega_1 \setminus \omega$. To find ξ, T and f as in the lemma, we use games.

For each $\alpha \in E^\lambda_{\omega_1}$ take a club $d_\alpha \subseteq \alpha$ of order-type ω_1, and let $\langle \delta^\alpha_\xi \mid \xi < \omega_1 \rangle$ be the increasing enumeration of d_α. Let $\vec{d} := \langle d_\alpha \mid \alpha \in E^\lambda_{\omega_1} \rangle$.

For each $\xi < \omega_1$ let $\Game(\xi)$ be the following two players game of length ω:

I	C_0	C_1	$\cdots$	C_n	$\cdots$
II	α_0	α_1	$\cdots$	α_n	$\cdots$

At the n-th stage first player I chooses a club $C_n \subseteq \mu_n^\xi$. Then II chooses $\alpha_n \in C_n$. If n is even, then II must chooses α_n so that $\alpha_n \in S_1^{\eta_n^\xi}$ if $\eta_n^\xi \bmod \omega \in b_\xi$ and so that $\alpha_n \in S_0^{\eta_n^\xi}$ if $\eta_n^\xi \bmod \omega \notin b_\xi$. Player II wins if

- $\mathrm{cl}_F(\xi \cup \{\alpha_n \mid n < \omega\}) \cap \omega_1 \subseteq \xi$,
- $\mathrm{cl}_F(\xi \cup \{\alpha_n \mid n < \omega\}) \cap \alpha_m \subseteq \delta_\xi^{\alpha_m}$ for all $m < \omega$.

Otherwise I wins.

Note that $\Game(\xi)$ is a closed game for II. Hence it is determined. We claim the following:

Claim. *There is $\xi \in \omega_1 \setminus \zeta$ such that II has a winning strategy for $\Game(\xi)$.*

Proof of Claim. Assume not. Then I has a winning strategy τ_ξ for $\Game(\xi)$ for each $\xi \in \omega_1 \setminus \zeta$. We may assume that τ_ξ is a function on ${}^{<\omega}(\lambda^+)$ such that $\tau_\xi(t)$ is a club subset of $\mu_{\mathrm{dom}(t)}^\xi$ for each $t \in {}^{<\omega}(\lambda^+)$. Let $\vec{\tau} := \langle \tau_\xi \mid \xi \in \omega_1 \setminus \zeta \rangle$.

Let θ be a sufficiently large regular cardinal, and take a countable $M \prec \langle \mathcal{H}_\theta, \in, \vec{\lambda}, \vec{S}, F, \zeta, \vec{d}, \vec{\tau} \rangle$. Let $\xi^* := M \cap \omega_1$. Note that

- $\bar{\sup}(M \cap \alpha) = \delta_{\xi^*}^\alpha$ for all $\alpha \in E_{\omega_1}^\lambda \cap M$

by the elementarity of M. For each $n < \omega$ let η_n^* and μ_n^* be $\eta_n^{\xi^*}$ and $\mu_n^{\xi^*}$, respectively.

By induction on $n < \omega$ we can take $\alpha_n \in M$ as follows:

- $\alpha_n \in \bigcap \{\tau_\xi(\langle \alpha_m \mid m < n \rangle) \mid \xi \in \omega_1 \setminus \zeta \ \wedge \ \mu_n^\xi = \mu_n^*\}$.
- Suppose that n is even. Then $\alpha_n \in S_1^{\eta_n^*}$ if $\eta_n^* \bmod \omega \in b_{\xi^*}$, and $\alpha_n \in S_0^{\eta_n^*}$ if $\eta_n^* \bmod \omega \notin b_{\xi^*}$.

We can take such $\alpha_n \in M$ because $\bigcap \{\tau_\xi(\langle \alpha_m \mid m < n \rangle) \mid \xi \in \omega_1 \setminus \zeta \ \wedge \ \mu_n^\xi = \mu_n^*\}$ is a club subset of μ_n^* which is in M, and $S_0^{\eta_n^*}$ and $S_1^{\eta_n^*}$ are stationary subsets of μ_n^* which are in M.

Let $C_n := \tau_{\xi^*}(\langle \alpha_m \mid m < n \rangle)$ for each $n < \omega$.

Then $\langle C_n, \alpha_n \mid n < \omega \rangle$ is a legal play in $\Game(\xi^*)$ in which I has moved according to the winning strategy τ_{ξ^*}. On the other hand, $x := \mathrm{cl}_F(\xi^* \cup \{\alpha_n \mid n < \omega\}) \subseteq M$ because $\xi^* \cup \{\alpha_n \mid n < \omega\} \subseteq M \prec \mathcal{M}$. Hence $x \cap \omega_1 = \xi^*$, and $x \cap \alpha_m \delta_{\xi^*}^{\alpha_m}$ for each even m. Thus II wins $\Game(\xi^*)$ with this play. This is a contradiction. $\square_{\text{Claim}}$

Fix $\xi \in \omega_1 \setminus \zeta$ such that II has a winning strategy τ for $\Game(\xi)$. Moreover let T' be the set of all $\langle \alpha_n \mid n < l \rangle$ $(l < \omega)$ which is a sequence II's moves according to τ against some I's moves $\langle C_n \mid n < l \rangle$. Then we can easily check that T' is a stationary $\vec{\mu}^\xi$-tree with the following properties:

- $\mathrm{cl}_F(\xi \cup \mathrm{ran}(t)) \cap \omega_1 = \xi$ for all $t \in T'$.
- For any $t \in T'$ and any even $n \in \mathrm{dom}(t)$ we have that $t(n) \in S_1^{\eta_n^\xi}$ if $\eta_n^\xi \bmod \omega \in b_\xi$ and that $t(n) \in S_0^{\eta_n^\xi}$ if $\eta_n^\xi \bmod \omega \notin b_\xi$.
- $\mathrm{cl}_F(\xi \cup \mathrm{ran}(t)) \cap t(n) \subseteq \delta_\xi^{t(n)}$ for all $t \in T'$ and all even $n \in \mathrm{dom}(t)$.

For each $t \in T'_{\mathrm{even}}$ by Fodor's lemma take $\delta_t < \mu^\xi_{\mathrm{dom}(t)}$ and a stationary $S_t \subseteq \mathrm{Suc}_{T'}(t)$ such that $\delta_\xi^\alpha = \delta_t$ for all $\alpha \in S_t$. Let T be the set of all $t \in T'$ such that $t(n) \in S_{t \restriction n}$ for all even $n \in \mathrm{dom}(t)$, and let f be the function on T_{even} such that $f(t) = \delta_t$. Then T and f are as desired. □

Clearly Lem.3.5 follows from Lem.3.7 together with Lem.3.8 below:

Lemma 3.8. *Let $k \leq \omega$ and, let $\vec{\mu} = \langle \mu_n \mid n < k \rangle$ be a sequence of regular cardinals $\geq \omega_2$ such that $(\sup\{\mu_m \mid \mu_m < \mu_n\})^\omega < \mu_n$ for all $n < k$. Suppose that T is an unbounded $\vec{\mu}$-tree, that $A \subseteq \{n < k \mid \forall m \in k \setminus \{n\},\ \mu_m \neq \mu_n\}$ and that f is a function on T_A with $f(t) < \mu_{\mathrm{dom}(t)}$. Then there are an unbounded $\vec{\mu}$-tree $\hat{T} \subseteq T$ and a function $\hat{f} \in \Pi_{n \in A} \mu_n$ such that $f(t) \leq \hat{f}(\mathrm{dom}(t))$ for all $t \in \hat{T}_A$.*

Before proving Lem.3.8 we need some preliminaries.

For $\vec{\mu} = \langle \mu_n \mid n < k \rangle$ and A as in Lem.3.8 let

$$A^- := \{n < k \mid \forall m \in A,\ \mu_n < \mu_m\}\ ,$$
$$A^+ := \{n < k \mid \forall m \in A,\ \mu_n > \mu_m\}\ .$$

Note that $A^- \,\dot\cup\, A \,\dot\cup A^+ = k$.

Lem.3.8 will be proved by induction on the order-type of $\{\mu_n \mid n < k\}$. In fact we prove a slightly stronger assertion, the following $(\sharp)_{\vec{\mu}}$, holds for every $\vec{\mu} = \langle \mu_n \mid n < k \rangle$ as in Lem.3.8:

$(\sharp)_{\vec{\mu}}$ For any T, A and f as in Lem.3.8 there are an unbounded $\vec{\mu}$-tree $\hat{T} \subseteq T$ and a function $\hat{f} \in \Pi_{n \in A} \mu_n$ such that

(i) $f(t) \leq \hat{f}(\mathrm{dom}(t))$ for all $t \in \hat{T}_A$,

(ii) $\mathrm{Suc}_{\hat{T}}(t) = \mathrm{Suc}_T(t)$ for all $t \in \hat{T}_{A^-}$.

The following lemma is a core of Lem.3.8:

Lemma 3.9. *Let k, $\vec{\mu}$, T, A and f be as in Lem.3.8. Suppose that $(\sharp)_{\vec{\nu}}$ holds for every sequence $\vec{\nu} = \langle \nu_n \mid n < l \rangle$ of regular cardinals $\geq \omega_2$ such that*

- $(\sup\{\nu_m \mid \nu_m < \nu_n\})^\omega < \nu_n$ *for all* $n < l$,
- $\text{o.t.}\{\nu_n \mid n < l\} \leq \text{o.t.}\{\mu_n \mid n \in A^- \cup A\}$.

Then there are an unbounded $\vec{\mu}$*-tree* $\hat{T} \subseteq T$ *and a function* $\hat{f} \in \Pi_{n\in A}\mu_n$ *such that*

(i) $f(t) \leq \hat{f}(\text{dom}(t))$ *for all* $t \in \hat{T}_A$,

(ii) $\text{Suc}_{\hat{T}}(t) = \text{Suc}_T(t)$ *for all* $t \in \hat{T}_{A^-}$.

Proof. We use games. For each $\hat{f} \in \Pi_{n\in A}\mu_n$ let $\Game(\hat{f})$ be the following two players game of length k:

I	β_0	β_1	$\cdots$	β_n	$\cdots$
II	α_0	α_1	$\cdots$	α_n	$\cdots$

Moves at the n-th stage depend on whether $n \in A^-$ or not. If $n \in A^-$, then I chooses $\beta_n \in \text{Suc}_T(\langle\alpha_m \mid m < n\rangle)$, and II plays $\alpha_n := \beta_n$. (So essentially II does nothing in this case.) If $n \notin A^-$, then I chooses $\beta_n < \mu_n$, and II chooses $\alpha_n \in \text{Suc}_T(\langle\alpha_m \mid m < n\rangle) \setminus \beta_n$. II wins if $f(\langle\alpha_m \mid m < n\rangle) \leq \hat{f}(n)$ for all $n \in A$. Otherwise I wins.

Note that $\Game(\hat{f})$ is a closed game for II. So $\Game(\hat{f})$ is determined.

Claim. *There is* $\hat{f} \in \Pi_{n\in A}\mu_n$ *such that II has a winning strategy for* $\Game(\hat{f})$.

Proof of Claim. Assume not. Then I has a winning strategy $\tau_{\hat{f}}$ for $\Game(\hat{f})$ for each $\hat{f} \in \Pi_{n\in A}\mu_n$.

Note that $\bar{\sup}\{\tau_{\hat{f}}(t) \mid \hat{f} \in \Pi_{n\in A}\mu_n\} < \mu_{\text{dom}(t)}$ for each $t \in T_{A^+}$ because $|\Pi_{n\in A}\mu_n| < \mu_{\text{dom}(t)}$ by the assumption on $\vec{\mu}$ in Lem.3.8. So for each $t \in T_{A^+}$ we can take

$$\alpha_t \in \text{Suc}_T(t) \setminus \bar{\sup}\{\tau_{\hat{f}}(t) \mid \hat{f} \in \Pi_{n\in A}\mu_n\} .$$

Let

$$T' := \{t \in T \mid \forall n \in \text{dom}(t) \cap A^+,\ t(n) = \alpha_{t\restriction n}\} .$$

So T' is a tree which does not branch at levels in A^+.

Next we take a collapse U of $T'_{A^-\cup A}$: Let $\sigma : \text{o.t.}(A^- \cup A) \to A^- \cup A$ be the increasing enumeration of $A^- \cup A$, and let

$$U := \{t \circ \sigma \mid t \in T'\} ,$$

where $t \circ \sigma$ denotes the function on $\sigma^{-1}[(A^- \cup A) \cap \mathrm{dom}(t)]$ such that $t \circ \sigma(m) = t(\sigma(m))$.

Let $\nu_m := \mu_{\sigma(m)}$ for $m < \text{o.t.}(A^- \cup A)$, and let $\vec{\nu} = \langle \nu_m \mid m < \text{o.t.}(A^- \cup A) \rangle$. Note that U is an unbounded $\vec{\nu}$-tree. Note also that $(\sharp)_{\vec{\nu}}$ holds by the assumption in the lemma.

Let $B := \sigma^{-1}[A]$ and $B^- := \sigma^{-1}[A^-]$. Moreover let g be the function on U_B such that $g(t \circ \sigma) = f(t)$ for each $t \in T'_A$.

Then by $(\sharp)_{\vec{\nu}}$ we can take an unbounded $\vec{\nu}$-tree $\hat{U} \subseteq U$ and a function $\hat{g} \in \Pi_{m \in B} \nu_m$ such that

(i) $g(u) \leq \hat{g}(\mathrm{dom}(u))$ for all $u \in \hat{U}_B$,

(ii) $\mathrm{Suc}_{\hat{U}}(u) = \mathrm{Suc}_U(u)$ for all $u \in \hat{U}_{B^-}$.

Let $T^* := \{t \in T' \mid t \circ \sigma \in \hat{U}\}$ and $\hat{f}^* := \hat{g} \circ \sigma^{-1} \in \Pi_{n \in A} \mu_n$. Then it is easy to see the following:

- $\mathrm{Suc}_{T^*}(t) = \{\alpha_t\}$ for all $t \in T^*_{A^+}$.
- $\mathrm{Suc}_{T^*}(t)$ is unbounded in $\mu_{\mathrm{dom}(t)}$ for all $t \in T^*_A$.
- $\mathrm{Suc}_{T^*}(t) = \mathrm{Suc}_T(t)$ for all $t \in T^*_{A^-}$.
- $f(t) \leq \hat{f}^*(\mathrm{dom}(t))$ for all $t \in T^*_A$.

Now by induction on $n < k$ we take $\beta_n < \mu_n$ and $\alpha_n \in \mathrm{Suc}_{T^*}(\langle \alpha_m \mid m < n \rangle)$. First let $\beta_n := \tau_{\hat{f}^*}(\langle \alpha_m \mid m < n \rangle)$. Then take α_n as follows:

- If $n \in A^-$, then let $\alpha_n := \beta_n$.
- If $n \in A$, then take $\alpha_n \in \mathrm{Suc}_{T^*}(\langle \alpha_m \mid m < n \rangle) \setminus \beta_n$.
- If $n \in A^+$, then let $\alpha_n := \alpha_{\langle \alpha_m \mid m < n \rangle}$.

From the observation on T^* in the previous paragraph we can take such $\alpha_n \in \mathrm{Suc}_{T^*}(\langle \alpha_m \mid m < n \rangle)$.

Here note that if $n \in A^+$, then

$$\alpha_n \;=\; \alpha_{\langle \alpha_m \mid m < n \rangle} \;\geq\; \bar{\sup}\{\tau_{\hat{f}}(\langle \alpha_m \mid m < n \rangle) \mid \hat{f} \in \Pi_{n \in A} \mu_n\} \;\geq\; \beta_n$$

by the choice of $\alpha_{\langle \alpha_m \mid m < n \rangle}$. Hence $\langle \beta_n, \alpha_n \mid n < k \rangle$ is a legal play of $\eth(\hat{f}^*)$ in which I has moved according to the winning strategy $\tau_{\hat{f}^*}$.

On the other hand, recall that $f(t) \leq \hat{f}^*(\mathrm{dom}(t))$ for all $t \in T^*_A$. Hence $f(\langle \alpha_m \mid m < n \rangle) \leq \hat{f}^*(n)$ for all $n \in A$, that is, II wins $\eth(\hat{f}^*)$ with $\langle \alpha_n \mid n < k \rangle$.

This is a contradiction. $\square_{\text{Claim}}$

Let $\hat{f} \in \Pi_{n\in A}\mu_n$ be such that II has a winning strategy τ for $\eth(\hat{f})$, and let $\hat{T}$ be the set of all $\langle \alpha_n \mid n < l\rangle$ ($l < k$ if $k = \omega$, and $l \leq k$ if $k < \omega$) which is a sequence of II's moves according to τ against some I's moves $\langle \beta_n \mid n < l\rangle$. Then it is easy to see that $\hat{T}$ and $\hat{f}$ are as desired. $\square$

Proof of Lem.3.8. By induction on o.t.$\{\mu_n \mid n < k\}$ we show that $(\sharp)_{\vec{\mu}}$ holds for all $\vec{\mu} = \langle \mu_n \mid n < k\rangle$ as in Lem.3.8. Note that if o.t.$\{\mu_n \mid n < k\} = 0$, then $(\sharp)_{\vec{\mu}}$ trivially holds.

(Successor step) Suppose that o.t.$\{\mu_n \mid n < k\}$ is successor and that T, A and f are as in Lem.3.8. Let $\mu^* := \max\{\mu_n \mid n < k\}$. If $\mu^* \notin \{\mu_n \mid n \in A\}$, then $(\sharp)_{\vec{\mu}}$ follows from the induction hypothesis and Lem.3.9.

Assume that $\mu^* \in \{\mu_n \mid n \in A\}$. Then there is a unique $n^* < k$ such that $\mu_{n^*} = \mu^*$. By the induction hypothesis and Lem.3.9 we can take an unbounded $\vec{\mu}$-tree $\hat{T} \subseteq T$ and $\hat{f}' \in \Pi_{n\in A\setminus\{n^*\}}\mu_n$ such that $f(t) \leq \hat{f}'(\mathrm{dom}(t))$ for all $t \in \hat{T}_{A\setminus\{n^*\}}$ and such that $\mathrm{Suc}_{\hat{T}}(t) = \mathrm{Suc}_T(t)$ for all $t \in \hat{T}_{A^-}$.

Here note that

$$|\hat{T}_{\{n^*\}}| = \max\{\mu_n \mid n < n^*\} < \mu_{n^*} \; .$$

So $\bar{\sup}\{f(t) \mid t \in \hat{T}_{\{n^*\}}\} < \mu_{n^*}$. Let $\hat{f} \in \Pi_{n\in A}\mu_n$ be an extension of $\hat{f}'$ such that $\hat{f}(n^*) = \bar{\sup}\{f(t) \mid t \in \hat{T}_{\{n^*\}}\}$. Then $\hat{T}$ and $\hat{f}$ are as desired.

(Limit step) Suppose that o.t.$\{\mu_n \mid n < k\}$ is limit and that T, A and f are as in Lem.3.8. Note that $k = \omega$. Take an increasing sequence $\langle \rho_m \mid m < \omega\rangle$ which converges to $\sup_{n<\omega}\mu_n$ and such that $\rho_0 = \min\{\mu_n \mid n \in A\}$. For each $m < \omega$ let $A_m := A \cap [\rho_m, \rho_{m+1})$.

Then, by the induction hypothesis and Lem.3.9, by induction on $m < \omega$ we can easily take T^m and f_m such that

(i) $T^0 = T$,

(ii) T^{m+1} is an unbounded $\vec{\mu}$-tree with $T^{m+1} \subseteq T^m$,

(iii) $f_m \in \Pi_{n\in A_m}\mu_n$,

(iv) $f(t) \leq f_m(\mathrm{dom}(t))$ for all $t \in (T^{m+1})_{A_m}$,

(v) $\mathrm{Suc}_{T^{m+1}}(t) = \mathrm{Suc}_{T^m}(t)$ for all $t \in (T^{m+1})_{A_m^-}$.

Let $\hat{T} := \bigcap_{m<\omega} T^m$ and $\hat{f} := \bigcup_{m<\omega} f_m$. Note that $\hat{T}$ is an unbounded $\vec{\mu}$-tree by (ii) and (v) above. Then it is easy to check that $\hat{T}$ and $\hat{f}$ are as desired. $\square$

References

[1] K. J. Devlin, *The Yorkshireman's guide to proper forcing*, Surveys in set theory, 60–115, London Math. Soc. Lecture Note Ser., 87, Cambridge Univ. Press, Cambridge, 1983.

[2] J. Cummings, M. Foreman and M. Magidor, *Square, scales and stationary reflection*, J. Math. Log. **1** (2001), no.1, 35–98.

[3] J. Cummings and M. Magidor, *Martin's Maximum and weak square*, Proc. Amer. Math. Soc. **139** (2011), no.9, 3339–3348.

[4] M. Foreman, M. Magidor and S. Shelah, *Martin's maximum, saturated ideals, and nonregular ultrafilters I*, Ann. of Math. **127** (1988), no.1, 1–47.

[5] S. D. Friedman and J. Krueger, *Thin stationary sets and disjoint club sequences*, Trans. Amer. Math. Soc. **359** (2007), 2407–2420.

[6] H. Sakai, *Stationary and semi-stationary reflection principle*, J. Symbolic Logic **73** (2008), no.1, 181–192.

[7] H. Sakai and B. Veličković, *Stationary reflection principles and two cardinal tree properties*, Journal of the Institute of Mathematics of Jussieu 14 (2015), no.1, pp.69–85.

[8] E. Schimmerling, *Combinatorial principles in the core model for one Woodin cardinal*, Ann. Pure Appl. Logic **74** (1995), no.2, 153–201.

[9] S. Shelah, *Proper and Improper Forcing*, Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1998.

[10] S. Shelah, *Cardinal Arithmetic*, Oxford Logic Guides 29, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1994.

[11] S. Todorčević, *Conjectures of Rado and Chang and cardinal arithmetic, in Finite and infinite combinatorics in sets and logic*, edited by N. Sauer, R. Woodrow, B. Sands, Kluwer Academic Publishers, 1993, 385-398.

[12] S. Todorčević and V. Torres, *Conjectures of Rado and Chang and special Aronszajn trees*, preprint.