Stationary reflection and 2^{ω_1}

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We show that the stationary reflection principle in the space $[\lambda]^\omega$ does not implies $2^{\omega_1} = \omega_2$. More precisely, we prove the following:

Definition 1. *For a cardinal $\lambda \geq \omega_2$, let $\mathsf{SR}(\lambda)$ be the following stationary reflection principle:*

$$\mathsf{SR}(\lambda) \equiv \text{For every stationary } S \subseteq [\lambda]^\omega \text{ there exists an } X \text{ such that } |X| = \omega_1 \subseteq X \text{ and } S \cap [X]^\omega \text{ is stationary in } [X]^\omega.$$

Theorem 2. *Suppose that there is a supercompact cardinal. Then there is a poset $\mathbb{P}$ which forces the following:*

(1) $\mathsf{SR}(\lambda)$ *holds for every cardinal* $\lambda \geq \omega_2$.

(2) $2^{\omega_1} = \omega_3$

Before starting the proof, we make some preliminaries. First we give notations on posets:

Notation 3. *For a regular cardinal γ and a set A of ordinals, let $\mathrm{Col}(\gamma, A)$ denote the Lévy collapse which adds surjections from γ to α for all $\alpha \in A$. Also, let $\mathrm{Add}(\gamma, A)$ denote the poset adding A-many new subsets of γ. More precisely,*

$$\mathrm{Col}(\gamma, A) = \text{the set of all partial functions } p \text{ on } \gamma \times A \text{ such that } |p| < \gamma \text{ and } p(\xi, \alpha) \in \alpha \text{ for every } \xi \in \gamma,$$

$$\mathrm{Add}(\gamma, A) = \text{the set of all partial functions } p : \gamma \times A \to 2 \text{ with } |p| < \gamma.$$

In both $\mathrm{Col}(\gamma, A)$ and $\mathrm{Add}(\gamma, A)$, the order is defined by reverse inclusions.

Next we review a standard fact on a sufficient condition for $\mathsf{SR}(\lambda)$:

Lemma 4. *Let λ be a cardinal $\geq \omega_2$ and assume that there is a proper forcing extension of V in which an elementary embedding $j : V \to M$ with the following properties is definable:*

(1) *M is a transitive model of* ZFC.

(2) $\mathrm{crit}(j) = {\omega_2}^V$ *and* $j({\omega_2}^V) > \lambda$.

(3) $j``\lambda \in M$.

Then $\mathsf{SR}(\lambda)$ *holds in* V.

Proof. In V, take an arbitrary stationary $S \subseteq [\lambda]^\omega$. Let W be a proper forcing extension of V and let $j : V \to M$ be an elementary embedding with the properties (1)-(3) which is definable in W. Working in W, we show that, in V, there is an X such that $|X| = \omega_1 \subseteq X$ and $S \cap [X]^\omega$ is stationary. Note that ω_1 is absolute among V, M and W.

First note that S remains stationary in $[\lambda]^\omega$ because W is a proper forcing extension of V. Hence $\{j``s \mid s \in S\}$ is stationary in $[j``\lambda]^\omega$. Here note that if $s \in S$ then $j``s = j(s)$ because $s$ is countable in $V$. Thus $\{j``s \mid s \in S\} \subseteq j(S) \cap [j``\lambda]^\omega$. Therefore $j(S) \cap [j``\lambda]^\omega$ is stationary in $[j``\lambda]^\omega$. This holds also in $M$. Moreover it follows from (2) that $|j``\lambda| = \omega_1 \subseteq j``\lambda$ in M. Hence, in M, there is an X such that $|X| = \omega_1 \subseteq X$ and $j(S) \cap [X]^\omega$ is stationary. Then, by the elementarity of j, in V, there is an X such that $|X| = \omega_1 \subseteq X$ and $S \cap [X]^\omega$ is stationary. $\square$

Now we prove Theorem 2:

Proof of Theorem 2. Suppose that κ is a supercompact cardinal. We show that $\mathrm{Col}(\omega_1, \kappa) \times \mathrm{Add}(\omega_1, \kappa^+)$ forces (1) and (2) of Theorem 2. Let $G \times H$ be a $\mathrm{Col}(\omega_1, \kappa) \times \mathrm{Add}(\omega_1, \kappa^+)$-generic filter over V. First of all, note that $\mathrm{Col}(\omega_1, \kappa) \times \mathrm{Add}(\omega_1, \kappa^+)$ has the κ-c.c. Hence, in $V[G][H]$, $\kappa = \omega_2$ and $(\kappa^+)^V = \omega_3$. Thus $2^{\omega_1} = \omega_3$ in $V[G][H]$. We show that, in $V[G][H]$, $\mathsf{SR}(\lambda)$ holds for every cardinal $\lambda \geq \kappa$.

Let $\lambda \geq \kappa$ be a cardinal in $V[G][H]$. Note that $\mathsf{SR}(\lambda)$ becomes stronger as λ becomes larger. Hence we may assume that $\lambda \geq (\kappa^+)^V$. In V, take a λ-supercompact embedding $j : V \to M$. Before proceeding, we make a remark here. Below, several models of ZFC will appear. But they are all σ-closed forcing extensions of V or M. Hence ω_1, $\mathrm{Col}(\omega_1, *)$ and $\mathrm{Add}(\omega_1, *)$ are absolute among them.

Consider $j(\mathrm{Col}(\omega_1, \kappa) \times \mathrm{Add}(\omega_1, \kappa^+))$. The following hold in both V and M:

- $j(\mathrm{Col}(\omega_1, \kappa) \times \mathrm{Add}(\omega_1, \kappa^+)) \cong$
  $\mathrm{Col}(\omega_1, \kappa) \times \mathrm{Col}(\omega_1, [\kappa, j(\kappa))) \times \mathrm{Add}(\omega_1, j``\kappa^+) \times \mathrm{Add}(\omega_1, j(\kappa^+) \setminus j``\kappa^+)$.
- $j \restriction \mathrm{Col}(\omega_1, \kappa) \times \mathrm{Add}(\omega_1, \kappa^+)$ is an isomorphism between $\mathrm{Col}(\omega_1, \kappa) \times \mathrm{Add}(\omega_1, \kappa^+)$ and $\mathrm{Col}(\omega_1, \kappa) \times \mathrm{Add}(\omega_1, j``\kappa^+)$.

(Note that all objects above belong to M because $\lambda \geq \kappa^+$ and $j : V \to M$ is a λ-supercompact embedding.) In particular, $G \times j``H$ is $\mathrm{Col}(\omega_1, \kappa) \times \mathrm{Add}(\omega_1, j``\kappa^+)$-generic over M.

Let $\bar{G} \times \bar{H}$ be a $\mathrm{Col}(\omega_1, [\kappa, j(\kappa))) \times \mathrm{Add}(\omega_1, j(\kappa^+) \setminus j``\kappa^+)$-generic filter over $V[G][H]$. Then $\bar{G} \times \bar{H}$ is also generic over $M[G][j``H]$. Moreover $p \in G \times H$ if and only if $j(p) \in G \times \bar{G} \times j``H \times \bar{H}$ for every $p \in \mathrm{Col}(\omega_1, \kappa) \times \mathrm{Add}(\omega_1, \kappa^+)$. Hence, in $V[G][H][\bar{G}][\bar{H}]$, the elementary embedding $j : V \to M$ can be extended to $j^* : V[G][H] \to M[G][\bar{G}][j``H][\bar{H}]$.

Now, $V[G][H][\bar{G}][\bar{H}]$ is a σ-closed forcing extension of $V[G][H]$. Moreover $j^* : V[G][H] \to M[G][\bar{G}][j``H][\bar{H}]$ satisfies the properties (1)-(3) in Lemma 4 for $V[G][H]$. Hence, by Lemma 4, $\mathsf{SR}(\lambda)$ holds in $V[G][H]$.

This completes the proof. $\square$