Notes on skinny stationary subsets of $\mathcal{P}_\kappa\lambda$

Hiroshi Sakai

1 Introduction

Definition 1.1 (Matsubara). *Let κ and λ be regular uncountable cardinals with $\kappa < \lambda$ and let S be a subset of $\mathcal{P}_\kappa\lambda$.*

(1) *S is said to be skinny if $|\{x \in S \mid \sup x = \alpha\}| < \alpha^{<\kappa}$ for every $\alpha < \lambda$.*

(2) *S is said to be skinnier if $|\{x \in S \mid \sup x = \alpha\}| \leq \alpha$ for every $\alpha < \lambda$.*

(3) *S is said to be skinniest if $|\{x \in S \mid \sup x = \alpha\}| \leq 1$ for every $\alpha < \lambda$.*

In this note, we consider the existence of stationary subsets of $\mathcal{P}_\kappa\lambda$ with these properties. First we give two situations in which skinniest stationary sets exist densely:

Proposition 1.2. *Assume that $V = L$. Let κ and λ be regular uncountable cardinals with $\kappa < \lambda$. Then for every stationary $T \subseteq \mathcal{P}_\kappa\lambda$, there exists a skinniest stationary $S \subseteq T$.*

Proposition 1.3. *Let κ and λ be regular uncountable cardinals with $\kappa < \lambda$. Suppose that δ is an inaccessible cardinal $> \lambda$. Then $\mathrm{Col}(\lambda, <\delta)$ forces that for every stationary $T \subseteq \mathcal{P}_\kappa\lambda$, there exists a skinniest stationary $S \subseteq T$. Here $\mathrm{Col}(\lambda, <\delta)$ denotes the Lévy collapse which makes δ to be λ^+.*

We also prove the following:

Proposition 1.4. *Let κ and λ be regular uncountable cardinals with $\kappa < \lambda$. Suppose that S is a skinnier stationary subset of $\mathcal{P}_\kappa\lambda$. Then $\mathrm{NS}_\lambda \restriction (\sup {}``S)$ is not λ^+-saturated.*

Hence if E is a stationary subset of λ consisting of ordinals of cofinality $< \kappa$ and $\mathrm{NS}_\lambda \restriction E$ is λ^+-saturated then $T := \{x \in \mathcal{P}_\kappa\lambda \mid \sup x \in E\}$ is stationary but there are no skinnier stationary $S \subseteq T$.

Proposition 1.2 and 1.3 are proved in Section 2. Proposition 1.4 is proved in Section 3.

2 Densely existence

Here we prove Proposition 1.2 and 1.3. Our argument goes through the following diamond principle:

Definition 2.1 (Matsubara)**.** *Let κ and λ be regular uncountable cardinals with $\kappa < \lambda$ and let T be a stationary subset of $\mathcal{P}_\kappa\lambda$. Then let $\diamondsuit_\lambda(T)$ be the following principle:*

$$\begin{aligned}\diamondsuit_\lambda(T) \ \equiv \ & \textit{there is a sequence } \langle b_\alpha \mid \alpha \in \sup``T\rangle \textit{ such that} \\ & \text{(i) } b_\alpha \subseteq \alpha \textit{ for every } \alpha \in \sup``T, \\ & \text{(ii) } \{x \in T \mid B \cap \sup x = b_{\sup x}\} \textit{ is stationary in } \mathcal{P}_\kappa\lambda.\end{aligned}$$

We call a sequence $\langle b_\alpha \mid \alpha \in \sup``T\rangle$ a $\diamondsuit_\lambda(T)$-sequence.

First we show that $\diamondsuit_\lambda(T)$ implies that T contains a skinniest stationary subset:

Lemma 2.2 (Matsubara)**.** *Let κ and λ be regular uncountable cardinals with $\kappa < \lambda$ and let T be a stationary subset of $\mathcal{P}_\kappa\lambda$. If $\diamondsuit_\lambda(T)$ holds then there exists a skinniest stationary $S \subseteq T$.*

Proof. We may assume that $x \cap \kappa \in \kappa$ and $\sup x \notin x$ for every $x \in T$. Assume that $\diamondsuit_\lambda(T)$ holds. Then we can easily take a sequence $\langle f_\alpha \mid \alpha \in \sup``T\rangle$ such that

(i) f_α is a function from ${}^{<\omega}\alpha$ to α for each $\alpha \in \sup``T$,

(ii) $\{x \in T \mid F \restriction {}^{<\omega}\sup x = f_{\sup x}\}$ is stationary for every $F : {}^{<\omega}\lambda \to \lambda$.

For each $\alpha \in \sup``T$, choose a $y_\alpha \in T$ such that $\sup y_\alpha = \alpha$ and $y_\alpha$ is closed under $f_\alpha$. If such $y_\alpha$ does not exist then let $y_\alpha$ be an arbitrary element of $T$ with $\sup y_\alpha = \alpha$. Then $S := \{y_\alpha \mid \alpha \in \sup``T\}$ is skinniest. We show that S is stationary.

Take an arbitrary function $F : {}^{<\omega}\lambda \to \lambda$. It suffices to find an $y \in S$ closed under F because $y \cap \kappa \in \kappa$ for every $y \in S$. By the property (ii) of $\langle f_\alpha \mid \alpha \in \sup``T\rangle$, there is an $x \in T$ such that $F \restriction {}^{<\omega}\sup x = f_{\sup x}$ and x is closed under F. Let $\alpha := \sup x$. Then, because x is closed under f_α, y_α is closed under f_α. But $f_\alpha = F \restriction {}^{<\omega}\alpha$ and $y_\alpha \subseteq \alpha$. Therefore y_α is closed under F. □

We show that, in both situations of Proposition 1.2 and 1.3, $\diamondsuit_\lambda(T)$ holds for every stationary $T \subseteq \mathcal{P}_\kappa\lambda$. By Lemma 2.2, this suffices. First we show this in the situation of Proposition 1.2:

Lemma 2.3. *Assume that $V = L$. Let κ and λ be regular uncountable cardinals with $\kappa < \lambda$. Then $\diamondsuit_\lambda(T)$ holds for every stationary $T \subseteq \mathcal{P}_\kappa\lambda$.*

Proof. Suppose that $T \subseteq \mathcal{P}_\kappa\lambda$ is stationary. By reducing T if necessary, we may assume that $\sup x \notin x$ and $x \cap \kappa \in \kappa$ for every $x \in T$. By induction on $\alpha \in \sup``T$, take a $b_\alpha \subseteq \alpha$ as follows. Assume that $\alpha \in \sup``T$ and that b_β has been taken for each $\beta \in \alpha \cap \sup``T$. Then let b_α be the $<_L$-least $b \subseteq \alpha$ with the following property:

$(*)_\alpha \equiv$ For some $f : {}^{<\omega}\alpha \to \alpha$, there are no $x \in T \cap L_\alpha$ such that $b \cap \sup x = b_{\sup x}$ and b is closed under f.

If such b does not exist then let $b_\alpha := \emptyset$. We show that $\langle b_\alpha \mid \alpha \in \sup``T\rangle$ is a $\diamondsuit_\lambda(T)$-sequence.

For the contradiction, assume that $\langle b_\alpha \mid \alpha \in \sup``T\rangle$ is not a $\diamondsuit_\lambda(T)$-sequence. Let B^* be the $<_L$-least $B \subseteq \lambda$ such that $\{x \in T \mid B^* \cap \sup x = b_{\sup x}\}$ is nonstationary. Then for some $F : {}^{<\omega}\lambda \to \lambda$, there are no $x \in T$ such that $B^* \cap \sup x = b_{\sup x}$ and x is closed under F. Here note that B^* is the $<_L$-least one for which such F exists. Let F^* be one of such F. Moreover let $\mathcal{M}$ be the structure $\langle L_{\lambda^+}, \in, T, \langle b_\alpha \mid \alpha \in \sup``T\rangle, B^*, F^*\rangle$.

Now, because T is stationary, $\{\sup x \mid x \in T \wedge x \text{ is closed under } F^*\}$ is stationary in λ. Hence there is an $x^* \in T$ such that x^* is closed under F^* and such that, letting M^* be the Skolem hull of $\sup x^*$ in $\mathcal{M}$, $M^* \cap \lambda = \sup x^*$. Then, by the standard argument using the transitive collapse of M^*, it is easy to see that $B^* \cap \sup x^*$ is the $<_L$-least $b \subseteq \sup x^*$ satisfying $(*)_{\sup x^*}$. Thus $b_{\sup x^*} = B^* \cap \sup x^*$ by the construction of $\langle b_\alpha \mid \alpha \in \sup``T\rangle$. Now $x^* \in T$, $B^* \cap \sup x^* = b_{\sup x^*}$ and x^* is closed under F^*. This contradicts the choice of B^* and F^*. □

Next we do in the situation of Proposition 1.3:

Lemma 2.4. *Let κ and λ be regular uncountable cardinals with $\kappa < \lambda$. Suppose that δ is an inaccessible cardinal. Then $\mathrm{Col}(\lambda, <\delta)$ forces that for every stationary $T \subseteq \mathcal{P}_\kappa\lambda$, $\diamondsuit_\lambda(T)$ holds.*

To show this, we prove two lemmata:

Lemma 2.5. *Suppose that κ and λ are regular uncountable cardinals such that $\kappa < \lambda$ and $\lambda^{<\kappa} = \lambda$. Let T be a stationary subset of $\mathcal{P}_\kappa\lambda$. Then $\mathrm{Add}(\lambda)$ forces that $\diamondsuit_\lambda(T)$ holds. Here $\mathrm{Add}(\lambda)$ denotes the poset ${}^{<\lambda}2$ ordered by reverse inclusion.*

Proof. Let $\mathbb{P}$ be the poset of all functions p such that $\mathrm{dom}(p) \in \lambda$ and $p(\alpha) \subseteq \alpha$ for every $\alpha \in \mathrm{dom}(p)$. The order is defined by reverse inclusion. It is easy to see that $\mathbb{P}$ is forcing equivalent with $\mathrm{Add}(\lambda)$. We show that if G is a $\mathbb{P}$-generic over V then $\langle \bigcup G\ (\alpha) \mid \alpha \in \sup``T\rangle$ is a $\diamondsuit_\lambda(T)$-sequence. We work in V. Let $\dot{G}$ be the canonical name for a $\mathbb{P}$-generic filter. We may assume that $\sup x \notin x$ and $x \cap \kappa \in \kappa$ for every $x \in T$.

Take an arbitrary $\mathbb{P}$-name $\dot{B}$ of a subset of λ, an arbitrary $\mathbb{P}$-name $\dot{F}$ of a function from ${}^{<\omega}\lambda$ to λ and an arbitrary $p \in \mathbb{P}$. We show that there is an $p^* \leq p$ and an $x^* \in T$ such that $p^* \Vdash$ "$\dot{B} \cap \sup x^* = \bigcup \dot{G}\ (\sup x^*)$ and x^* is closed under $\dot{F}$".

By induction on $\alpha \in \lambda$, we can construct a descending sequence $\langle p_\alpha \mid \alpha \in \lambda\rangle$ below p such that p_α decides $\dot{B} \cap \alpha$ and $\dot{F} \restriction {}^{<\omega}\alpha$ for each $\alpha \in \lambda$. Let $B \subseteq \lambda$ and $F : {}^{<\omega}\lambda \to \lambda$ be the evaluations of $\dot{B}$ and $\dot{F}$ by $\langle p_\alpha \mid \alpha \in \lambda\rangle$. That is,

- $B = \{\beta \in \lambda \mid (\exists \alpha \in \lambda)\ p_\alpha \Vdash \text{"}\beta \in \dot{B}\text{"}\}$,
- $F(a) = \beta$ if and only if $p_\alpha \Vdash$ "$\dot{F}(a) = \beta$" for some $\alpha \in \lambda$.

Now, because T is stationary, there is an $x^* \in T$ such that x^* is closed under F and $\mathrm{dom}(p_\alpha) < \sup x^*$ for every $\alpha < \sup x^*$. Let $p' := \bigcup_{\alpha < \sup x^*} p_\alpha$. First note that $p' \Vdash$ "$\dot{F} \restriction {}^{<\omega} \sup x^* = F \restriction {}^{<\omega} \sup x^*$". Hence $p' \Vdash$ "x^* is closed under $\dot{F}$". Note also that $p' \Vdash$ "$\dot{B} \cap \sup x^* = B \cap \sup x^*$" and that $\mathrm{dom}(p') \subseteq \sup x^*$. Take an $p^* \leq p'$ such that $p^*(\sup x^*) = B \cap \sup x^*$. Then $p^* \Vdash$ "$\dot{B} \cap \sup x = \bigcup \dot{G}\ (\sup x^*)$". Therefore p^* and x^* are those desired. $\square$

Although the second one can be proved by the similar argument as the first one, we give proof.

Lemma 2.6. *Let κ and λ be regular uncountable cardinals with $\kappa < \lambda$. Suppose that T is a stationary subset of $\mathcal{P}_\kappa \lambda$ such that $\diamondsuit_\lambda(T)$ holds. Then every λ-closed forcing preserves $\diamondsuit_\lambda(T)$.*

Proof. Let $\langle b_\alpha \mid \alpha \in \sup``T \rangle$ be a $\diamondsuit_\lambda(T)$-sequence. Take an arbitrary $\lambda$-closed poset $\mathbb{P}$. We show that $\langle b_\alpha \mid \alpha \in \sup``T \rangle$ remains to be a $\diamondsuit_\lambda(T)$-sequence in the generic extension by $\mathbb{P}$. We may assume that $\sup x \notin x$ and $x \cap \kappa \in \kappa$ for each $x \in T$.

Take an arbitrary $\mathbb{P}$-name $\dot{B}$ of a subset of λ, an arbitrary $\mathbb{P}$-name $\dot{F}$ of a function from ${}^{<\omega}\lambda$ to λ and an arbitrary $p \in \mathbb{P}$. It suffices to find a $p^* \leq p$ and an $x^* \in T$ such that $p^* \Vdash_{\mathbb{P}}$ "$\dot{B} \cap \sup x^* = b_{\sup x^*}$ and x^* is closed under $\dot{F}$".

By induction on $\alpha < \lambda$, we can construct a descending sequence $\langle p_\alpha \mid \alpha \in \lambda \rangle$ such that p_α decides $\dot{B} \cap \alpha$ and $\dot{F} \restriction {}^{<\omega}\alpha$. Let B and F be the evaluations of $\dot{B}$ and $\dot{F}$ by $\langle p_\alpha \mid \alpha \in \lambda \rangle$.

Then, because $\langle b_\alpha \mid \alpha \in \sup``T \rangle$ is a $\diamondsuit_\lambda(T)$-sequence, there is an $x^* \in T$ such that $B \cap \sup x^* = b_{\sup x^*}$ and x^* is closed under F. Let $p^* := p_{\sup x^*}$. Then $p^* \Vdash$ "$\dot{B} \cap \sup x^* = B \cap \sup x^*$ and $\dot{F} \restriction {}^{<\omega} \sup x^* = F \restriction {}^{<\omega} \sup x^*$". Hence $p^* \Vdash$ "$\dot{B} \cap \sup x^* = b_{\sup x^*}$ and x^* is closed under $\dot{F}$". Thus p^* and x^* are those desired. $\square$

Lemma 2.4 easily follows from the above two lemmata:

Proof of Lemma 2.4. Suppose that G is a $\mathrm{Col}(\lambda, <\delta)$-generic filter over V. In $V[G]$, take an arbitrary stationary $T \subseteq \mathcal{P}_\kappa \lambda$. Then $T \in V[G \cap \mathrm{Col}(\lambda, <\gamma)]$ for some $\gamma < \lambda$. Let G_0 denotes $G \cap \mathrm{Col}(\lambda, <\gamma)$. Note that, in V, $\mathrm{Col}(\lambda, <\delta)$ is forcing equivalent with $\mathrm{Col}(\lambda, <\gamma) \times \mathrm{Add}(\lambda) \times \mathrm{Col}(\lambda, <\delta)$. Note also that $\mathrm{Add}(\lambda)$ and $\mathrm{Col}(\lambda, <\delta)$ are absolute among all models between V and $V[G]$.

Take an $\mathrm{Add}(\lambda)$-generic filter G_1 and a $\mathrm{Col}(\lambda, <\delta)$-generic filter G_2 such that $V[G] = V[G_0][G_1][G_2]$. Then $\diamondsuit_\lambda(T)$ holds in $V[G_0][G_1]$ by Lemma 2.5. Moreover, because $\mathrm{Col}(\lambda, <\delta)$ is λ-closed, $\diamondsuit_\lambda(T)$ still holds in $V[G_0][G_1][G_2] = V[G]$ by Lemma 2.6. $\square$

3 Nonsaturation of suprema

Here we prove Proposition 1.4. Our argument goes through a combinatorial principle obtained from weakening the ♣-principle. First we introduce this combinatorial principle. Below, for regular uncountable cardinals $\kappa < \lambda$,

$$E^{\lambda}_{<\kappa} \;:=\; \{\alpha \in \lambda \mid \mathrm{cf}(\alpha) < \kappa\}\ .$$

Definition 3.1. *Suppose that κ and λ are regular uncountable cardinals with $\kappa < \lambda$ and that $E \subseteq E^{\lambda}_{<\kappa}$ is stationary. Then let $\spadesuit_{\lambda,<\kappa}(E)$ and $\spadesuit^{-}_{\lambda,<\kappa}(E)$ be the following principles:*

$\spadesuit_{\lambda,<\kappa}(E) \;\equiv\;$ *There is a sequence $\langle b_\alpha \mid \alpha \in E\rangle$ such that*

(i) *$b_\alpha \in \mathcal{P}_\kappa\alpha$ for each $\alpha \in E$,*

(ii) *$\{\alpha \in E \mid B \cap b_\alpha$ is unbounded in $\alpha\}$ is stationary for every unbounded $B \subseteq \lambda$.*

$\spadesuit^{-}_{\lambda,<\kappa}(E) \;\equiv\;$ *There is a sequence $\langle \mathfrak{b}_\alpha \mid \alpha \in E\rangle$ such that*

(i') *$\mathfrak{b}_\alpha \subseteq \mathcal{P}_\kappa\alpha$ and $|\mathfrak{b}_\alpha| \leq \alpha$ for every $\alpha \in E$,*

(ii') *$\{\alpha \in E \mid (\exists b \in \mathfrak{b}_\alpha)\ B \cap b$ is unbounded in $\alpha\}$ is stationary for every unbounded $B \subseteq \lambda$.*

We call a sequence $\langle b_\alpha \mid \alpha \in E\rangle$ satisfying (i) *and* (ii) *a $\spadesuit_{\lambda,<\kappa}(E)$-sequence. Also we call a sequence $\langle \mathfrak{b}_\alpha \mid \alpha \in E\rangle$ satisfying* (i') *and* (ii') *a $\spadesuit^{-}_{\lambda,<\kappa}(E)$-sequence.*

We only use $\spadesuit^{-}_{\lambda,<\kappa}$. The following is easy:

Lemma 3.2. *Suppose that κ and λ are regular uncountable cardinals with $\kappa < \lambda$ and that $S \subseteq \mathcal{P}_\kappa\lambda$ is skinnier stationary. Then $\spadesuit^{-}_{\lambda,<\kappa}(\sup``S)$ holds.*

Proof. We may assume that $\sup x \notin x$ for every $x \in S$. For each $\alpha \in \sup``S$, let $\mathfrak{b}_\alpha$ the set of all $x \in S$ with $\sup x = \alpha$. Then the sequence $\langle \mathfrak{b}_\alpha \mid \alpha \in \sup``S\rangle$ satisfies (i'). We show that this sequence satisfies (ii').

Take an arbitrary unbounded $B \subseteq \lambda$. Let $F : \lambda \to \lambda$ be such that $F(\alpha) = \min(B \setminus (\alpha + 1))$ and let S' be the set of all $x \in S$ closed under F. Note that if $x \in \mathcal{P}_\kappa\lambda$ is closed under F then $B \cap \sup x$ is unbounded in $\sup x$. Hence $\sup``S' \subseteq \{\alpha \in E \mid (\exists b \in \mathfrak{b}_\alpha)\ B\cap b$ is unbounded in $\alpha\}$. But $\sup``S'$ is stationary in λ because S' is stationary in $\mathcal{P}_\kappa\lambda$. Therefore $\{\alpha \in E \mid (\exists b \in \mathfrak{b}_\alpha)\ B \cap b$ is unbounded in $\alpha\}$ is stationary in λ. □

Now the following suffices for Proposition 1.4:

Lemma 3.3. *Suppose that κ and λ are regular uncountable cardinals with $\kappa < \lambda$ and that $E \subseteq E^{\lambda}_{<\kappa}$ is stationary. If $\spadesuit^{-}_{\lambda,<\kappa}(E)$ holds then $\mathrm{NS}_\lambda \restriction E$ is not λ^+-saturated.*

Our proof of this lemma is based on that of Shelah's well-known theorem that there are no ω_3-saturated normal ideal over ω_2 concentrating on $E^{\omega_2}_{\omega}$. As is Shelah's theorem, a key object of proof of Lemma 3.3 is a strongly pairwise almost disjoint family. First recall the notion of strongly pairwise almost disjoint family:

Definition 3.4. *Let λ be a limit ordinal and let $\mathcal{B}$ be a family of unbounded subsets of λ. $\mathcal{B}$ is said to be strongly pairwise almost disjoint if for every $\mathcal{B}' \subseteq \mathcal{B}$ with $|\mathcal{B}'| \leq \lambda$, there is a $\sigma : \mathcal{B}' \to \lambda$ such that $B_1 \cap B_2 \subseteq \max\{\sigma(B_1), \sigma(B_2)\}$ for each distinct $B_1, B_2 \in \mathcal{B}'$.*

We prove two lemmata on a strongly pairwise almost disjoint family. The first one is standard:

Lemma 3.5. *Let λ be a regular cardinal. Then there is a strongly pairwise almost disjoint family of unbounded subsets of λ which has the size λ^+.*

Proof. By the standard argument, we can take a pairwise almost disjoint family $\mathcal{B}$ of unbounded subsets of λ with $|\mathcal{B}| = \lambda^+$. We show that $\mathcal{B}$ is strongly pairwise almost disjoint.

Take an arbitrary $\mathcal{B}' \subseteq \mathcal{B}$ with $|\mathcal{B}'| \leq \lambda$. Let $\langle B_\xi \mid \xi < |\mathcal{B}'| \rangle$ be a 1-1 enumeration of $\mathcal{B}'$. For each $\xi < |\mathcal{B}'|$, let $\sigma(B_\xi) := \sup\{\sup(B_\xi \cap B_\eta) \mid \eta < \xi\}$. Note that if $\xi < |\mathcal{B}'|$ then $\sigma(B_\xi) < \lambda$ by the regularity of λ. Now, clearly, σ is a witness for $\mathcal{B}'$ of that $\mathcal{B}$ is strongly pairwise almost disjoint. □

The second one is easy but is a key. In the following, the only interesting case is when $\mathrm{cf}(\lambda) < |\lambda|$ and b is an unbounded subset of λ with $|b| < |\lambda|$.

Lemma 3.6. *Suppose that λ is a limit ordinal and that $\mathcal{B}$ is a strongly pairwise almost disjoint family of unbounded subsets of λ. Let b be a subset of λ with $|b|^+ \leq \lambda$. Then*

$$|\{B \in \mathcal{B} \mid B \cap b \textit{ is unbounded in } \lambda\}| \;\leq\; |b| \,.$$

Proof. For the contradiction, assume not. Then b is an unbounded subset of λ and hence $\mathrm{cf}(\lambda) \leq |b|$. Take a $\mathcal{B}' \subseteq \{B \in \mathcal{B} \mid B \cap b \text{ is unbounded in } \lambda\}$ with $|\mathcal{B}'| = |b|^+$. Let $\sigma : \mathcal{B}' \to \lambda$ be a witness of that $\mathcal{B}$ is strongly pairwise almost disjoint.

Then there is an $\alpha^* < \lambda$ such that $|\sigma^{-1}``\alpha^*| = |b|^+$ because $\mathrm{cf}(\lambda) < |b|^+$. Let $\mathcal{B}^* := \{B \setminus \alpha^* \mid B \in \mathcal{B}' \wedge \sigma(B) < \alpha^*\}$. Then $\mathcal{B}^*$ is pairwise disjoint, $|\mathcal{B}^*| = |b|^+$ and b intersects with every member of $\mathcal{B}^*$. This is a contradiction. □

Now we can prove Lemma 3.3 easily:

Proof of Lemma 3.3. For the contradiction, assume that $\spadesuit^-_{\lambda,<\kappa}(E)$ holds and that $\mathrm{NS}_\lambda \restriction E$ is λ^+-saturated. Take a $\spadesuit^-_{\lambda,<\kappa}(E)$-sequence $\langle \mathfrak{b}_\alpha \mid \alpha \in E \rangle$. Let $\mathbb{P}$ denote the corresponding poset of $\mathrm{NS}_\lambda \restriction E$, that is, the poset of all stationary subsets of E ordered by inclusion. Moreover, by Lemma 3.5, take a strongly pairwise almost disjoint family $\mathcal{B}$ of unbounded subsets of λ such that $|\mathcal{B}| = \lambda^+$.

Let $\dot{G}$ be the canonical name for $\mathbb{P}$-generic filter and, in $V^{\mathbb{P}}$, let $j : V \to M \cong \mathrm{Ult}(V, \dot{G})$ be the generic elementary embedding. Moreover let $\dot{\mathfrak{b}}_\lambda$ be a $\mathbb{P}$-name for the λ-th element of $j(\langle \mathfrak{b}_\alpha \mid \alpha \in E \rangle)$ and let $\dot{\mathcal{A}}$ be a $\mathbb{P}$-name of the set $\{B \in \mathcal{B} \mid (\exists b \in \dot{\mathfrak{b}}_\lambda)\; B \cap b \text{ is unbounded in } \lambda\}$. Here note that $\mathbb{P}$ forces the following:

- $\mathcal{B}$ remains to be strongly pairwise almost disjoint.
- κ remains to be a regular cardinal and $\dot{\mathfrak{b}}_\lambda \subseteq \mathcal{P}_\kappa \lambda$. In particular, $|b|^+ \leq \kappa < \lambda$ for every $b \in \dot{\mathfrak{b}}_\lambda$.
- $|\dot{\mathfrak{b}}_\lambda| \leq \lambda$.

(The first one follows from the fact that $\mathbb{P}$ has the λ^+-c.c.) Hence $\Vdash_{\mathbb{P}}$ " $|\dot{\mathcal{A}}| \leq \lambda$ " by Lemma 3.6. Then, because $\mathbb{P}$ has the λ^+-c.c., we can take an $\mathcal{A}^* \in V$ such that $|\mathcal{A}^*| = \lambda$ and $\Vdash_{\mathbb{P}}$ " $\dot{\mathcal{A}} \subseteq \mathcal{A}^*$ ".

Take a $B \in \mathcal{B} \setminus \mathcal{A}^*$. Then

$$\Vdash_{\mathbb{P}} \text{“} (\forall b \in \dot{\mathfrak{b}}_\lambda)\ B \cap b \text{ is bounded in } \lambda \text{”} .$$

On the other hand, $E^* := \{\alpha \in E \mid (\exists b \in \mathfrak{b}_\alpha)\ B \cap b \text{ is unbounded in } \alpha\}$ is stationary and

$$E^* \ \Vdash_{\mathbb{P}} \text{“} (\exists b \in \dot{\mathfrak{b}}_\lambda)\ j(B) \cap b \text{ is unbounded in } \lambda \text{”} .$$

But $\Vdash_{\mathbb{P}}$ " $j(B) \cap \lambda = B$ ". Hence

$$E^* \ \Vdash_{\mathbb{P}} \text{“} (\exists b \in \dot{\mathfrak{b}}_\lambda)\ B \cap b \text{ is unbounded in } \lambda \text{”} .$$

This is a contradiction. □