Note on least singular cardinal with $\mathrm{cf}([\lambda]^{\mathrm{cf}(\lambda)}, \subseteq) > \lambda^+$

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1 Introduction

The Singular Cardinal Hypothesis (SCH) is a restriction of the Generalized Continuum Hypothesis (GCH) to singular cardinals, and it has been extensively studied by set theorists. First recall SCH:

$$\mathsf{SCH} \;\equiv\; \lambda^{\mathrm{cf}(\lambda)} = \lambda^+ \text{ for all singular cardinals } \lambda \text{ with } 2^{\mathrm{cf}(\lambda)} < \lambda.$$

Note that for a singular cardinal λ it holds that

$$\lambda^{\mathrm{cf}(\lambda)} \;=\; \mathrm{cf}([\lambda]^{\mathrm{cf}(\lambda)}, \subseteq) \cdot 2^{\mathrm{cf}(\lambda)} \;,$$

where for a directed set $\langle D, < \rangle$

$$\mathrm{cf}(D, <) \;:=\; \min\{|X| \mid X \text{ is a } <\text{-cofinal subset of } D\} \;.$$

Thus the following SCH^+ implies SCH:

$$\mathsf{SCH}^+ \;\equiv\; \mathrm{cf}([\lambda]^{\mathrm{cf}(\lambda)}, \subseteq) = \lambda^+ \text{ for all singular cardinals } \lambda.$$

In this note we review three basic theorems on the failure of SCH^+ due to Shelah [3]. The first one is a variant of Silver's theorem for SCH^+:

Theorem 1.1 (Shelah [3]). *Assume that SCH^+ fails, and let λ be the least singular cardinal at which SCH^+ fails. Then $\mathrm{cf}(\lambda) = \omega$.*

The second one is on the pseudo power. For the definition of the pseudo power of λ, denoted as $\mathrm{pp}(\lambda)$, see Section 2.1:

Theorem 1.2 (Shelah [3]). *Assume that SCH^+ fails, and let λ be the least singular cardinal at which SCH^+ fails. Then $\mathrm{pp}(\lambda) > \lambda^+$.*

The last one is on a useful combinatorial consequence of the failure of SCH^+. For definition of a better scale see Section 2.1:

Theorem 1.3 (Shelah [3]). *Assume that* SCH^+ *fails, and let* λ *be the least singular cardinal at which* SCH^+ *fails. Then there exists a set* A *of regular cardinals such that*

(i) $\sup(A) = \lambda$, *and* $\text{o.t.}(A) = \omega$,

(ii) $\langle \Pi A, <^* \rangle$ *has a better scale of length* λ^+.

From the above theorems we have the following corollary:

Corollary 1.4. *Assume that* SCH *fails. Then there exists a singular cardinal* λ *with the following properties:*

(i) $\text{cf}(\lambda) = \omega$.

(ii) $\text{pp}(\lambda) > \lambda^+$.

(iii) There exists a set A *of regular cardinals such that* $\sup(A) = \lambda$, *such that* $\text{o.t.}(A) = \omega$ *and such that* $\langle \Pi A, <^* \rangle$ *has a better scale of length* λ^+.

Thm.1.1, 1.2 and 1.3 will be proved in Section 3, 4 and 5, respectively. In Section 2 we review basics on PCF theory used in this note. The author referred to Abraham-Magidor [1] for Section 2.

At the end of this section we present miscellaneous notation used in this note. See Section 2.1 for notation and basic definitions in PCF theory.

Let A be a set of ordinals. Then $\text{o.t.}(A)$ denotes the order-type of A. Moreover $\text{Lim}(A)$ denotes the set of all limit points in A, that is, the set $\{\alpha \in \text{On} \mid \sup(A \cap \alpha) = \alpha\}$.

For regular cardinals $\mu < \nu$ let

$$\begin{aligned} E^\nu_\mu &:= \{\alpha < \nu \mid \text{cf}(\alpha) = \mu\} \\ E^\nu_{<\mu} &:= \{\alpha < \nu \mid \text{cf}(\alpha) < \mu\} \\ E^\nu_{>\mu} &:= \{\alpha < \nu \mid \text{cf}(\alpha) > \mu\} \end{aligned}$$

Suppose that $\mathcal{M}$ is a structure on which a well-ordering of its universe is definable. Then for $A \subseteq M$ let $\text{Sk}^{\mathcal{M}}(A)$ denote the Skolem hull of A in $\mathcal{M}$, i.e. the smallest $M \prec \mathcal{M}$ with $A \subseteq M$.

Let μ be a limit ordinal. Then a set x is said to be *internally approachable of length* μ if there exists a $\subseteq$-increasing sequence $\langle x_\xi \mid \xi < \mu \rangle$ such that

- $\bigcup_{\xi<\mu} x_\xi = x$,
- $\langle x_\xi \mid \xi < \zeta \rangle \in x$ for all $\zeta < \mu$.

A sequence $\langle x_\xi \mid \xi < \mu \rangle$ as above is called an *internally approaching sequence* to x.

2 Basics in PCF theory

2.1 Notation and basic definitions

Here we give notation and basic definitions in PCF theory.

Let A be a set of cardinals. For a set $\mathcal{F} \subseteq {}^A\mathrm{On}$ let $\sup(\mathcal{F}) \in {}^A\mathrm{On}$ be such that

$$\sup(\mathcal{F})(\kappa) = \sup\{f(\kappa) \mid f \in \mathcal{F}\}$$

for each $\kappa \in A$. Next let $f, g \in {}^A\mathrm{On}$. Then let

$$f < g \quad \overset{\text{def}}{\Leftrightarrow} \quad \forall\kappa \in A,\ f(\kappa) < g(\kappa)\ ,$$
$$f \leq g \quad \overset{\text{def}}{\Leftrightarrow} \quad \forall\kappa \in A,\ f(\kappa) \leq g(\kappa)\ .$$

For an ideal I over A let

$$f <_I g \quad \overset{\text{def}}{\Leftrightarrow} \quad \{\kappa \in A \mid f(\kappa) \not< g(\kappa)\} \in I,$$
$$f \leq_I g \quad \overset{\text{def}}{\Leftrightarrow} \quad \{\kappa \in A \mid f(\kappa) \not\leq g(\kappa)\} \in I\ ,$$
$$f =_I g \quad \overset{\text{def}}{\Leftrightarrow} \quad \{\kappa \in A \mid f(\kappa) \neq g(\kappa)\} \in I\ .$$

For an ordinal ν let

$$f <_\nu g \quad \overset{\text{def}}{\Leftrightarrow} \quad \forall\kappa \in A \setminus \nu,\ f(\kappa) < g(\kappa)\ ,$$
$$f \leq_\nu g \quad \overset{\text{def}}{\Leftrightarrow} \quad \forall\kappa \in A \setminus \nu,\ f(\kappa) \leq g(\kappa)\ ,$$
$$f =_\nu g \quad \overset{\text{def}}{\Leftrightarrow} \quad \forall\kappa \in A \setminus \nu,\ f(\kappa) = g(\kappa)\ .$$

Finally let

$$f <^* g \quad \overset{\text{def}}{\Leftrightarrow} \quad \exists\nu < \sup(A),\ f <_\nu g\ ,$$
$$f \leq^* g \quad \overset{\text{def}}{\Leftrightarrow} \quad \exists\nu < \sup(A),\ f \leq_\nu g\ ,$$
$$f =^* g \quad \overset{\text{def}}{\Leftrightarrow} \quad \exists\nu < \sup(A),\ f =_\nu g\ .$$

Note that $<^*$, $\leq^*$ and $=^*$ coincide $<_I$, $\leq_I$ and $=_I$ for the bounded ideal I over A, respectively.

Definition 2.1. *Let A be a set of cardinals and I be an ideal over A. Suppose that $\mathcal{F}$ is a subset of ${}^A\mathrm{On}$. Then $g \in {}^A\mathrm{On}$ is said to be* the exact upper bound *of $\mathcal{F}$ with respect to $<_I$ if*

(i) $f <_I g$ for all $f \in \mathcal{F}$,

(ii) for any $h \in {}^A\mathrm{On}$ with $h <_I g$ there exists $f \in \mathcal{F}$ with $h <_I f$.

Note that $\mathcal{F}$ may not have the exact upper bound with respect to $<_I$. Note also that the exact upper bound of $\mathcal{F}$ with respect to $<_I$ is unique modulo I if it exists.

Definition 2.2. *Let A be a set of cardinals and I be an ideal over A. A* scale *in $\langle \Pi A, <_I \rangle$ is a $<_I$-increasing $<_I$-cofinal sequence in ΠA whose length is a regular cardinal.*

Note that if $\vec{f} = \langle f_\alpha \mid \alpha < \nu \rangle$ is a scale in $\langle \Pi A, <_I \rangle$, then the identity function on A is the exact upper bound of $\vec{f}$ with respect to $<_I$. In general there may not be any scale in $\langle \Pi A, <_I \rangle$. But if I is a maximal ideal, then $\langle \Pi A, <_I \rangle$ has a scale because it is linear.

Note also that if $\langle \Pi A, <_I \rangle$ has scales, then all scales in $\langle \Pi A, <_I \rangle$ have the same length. Moreover if λ is a singular cardinal, A is a set of regular cardinals with $\sup(A) = \lambda$, and I is an ideal over A including the bounded ideal, then $\langle \Pi A, <_I \rangle$ is $< \lambda^+$-directed. Thus the length of a scale in $\langle \Pi A, <_I \rangle$ must be greater than or equal to λ^+ for such A and I.

Definition 2.3. *Let A be a set of cardinals and I be an ideal over A. Suppose that $\langle \Pi A, <_I \rangle$ has a scale. Then let*

$$\mathrm{tcf}(\Pi A, <_I) \ := \ \textit{the length of scales in } \langle \Pi A, <_I \rangle.$$

$\mathrm{tcf}(\Pi A, <_I)$ *is called* the true cofinality *of $\langle \Pi A, <_I \rangle$.*

Definition 2.4. *Let λ be a singular cardinal, and let Ω_λ be the set of all pairs $\langle A, I \rangle$ such that*

(i) A is a set of regular cardinals with $\sup(A) = \lambda$ and $\mathrm{o.t.}(A) = \mathrm{cf}(\lambda)$,

(ii) I is a maximal ideal over A including the bounded ideal.

Then let

$$\mathrm{pp}(\lambda) \ := \ \sup\{\mathrm{tcf}(\Pi A, I) \mid \langle A, I \rangle \in \Omega_\lambda\} \ .$$

$\mathrm{pp}(\lambda)$ *is called* the pseudo power *of λ.*

Note that $\mathrm{pp}(\lambda) \geq \lambda^+$ by the remark before Def.2.3. Note also the following:

Proposition 2.5. $\mathrm{pp}(\lambda) \leq \mathrm{cf}([\lambda]^{\mathrm{cf}(\lambda)}, \subseteq)$ *for all singular cardinal λ.*

Proof. Suppose that A is a set of regular cardinals with $\sup(A) = \lambda$ and $\mathrm{o.t.}(A) = \mathrm{cf}(\lambda)$ and that I is a maximal ideal over A including the bounded ideal. We prove that $\mathrm{tcf}(\Pi A, <_I) \leq \mathrm{cf}([\lambda]^{\mathrm{cf}(\lambda)}, \subseteq)$. We may assume that $\min(A) > \mathrm{cf}(\lambda)$.

Take a $\subseteq$-cofinal $X \subseteq [\lambda]^{\mathrm{cf}(\lambda)}$ with $|X| = \mathrm{cf}([\lambda]^{\mathrm{cf}(\lambda)}, \subseteq)$. For each $x \in X$ define $f_x \in \Pi A$ as $f_x(\kappa) = \sup(x \cap \kappa)$ for each $\kappa \in A$. Note that for any $g \in \Pi A$ if we take $x \in X$ with $g[A] \subseteq x$, then $g \leq f_x$. Hence $\{f_x \mid x \in X\}$ is $<_I$-cofinal in ΠA. Therefore $\mathrm{tcf}(\Pi A, <_I) \leq |X| = \mathrm{cf}([\lambda]^{\mathrm{cf}(\lambda)}, \subseteq)$. □

Next we give the notion of good and better scales. We are interested in only good and better scales in $\langle \Pi A, <^* \rangle$ of length $\sup(A)^+$:

Definition 2.6. *Let λ be a singular cardinal and A be a set of regular cardinals with $\sup(A) = \lambda$. Moreover let $\vec{f} = \langle f_\alpha \mid \alpha < \lambda^+ \rangle$ be a $<^*$-increasing sequence in ${}^A\mathrm{On}$.*

(1) $\alpha \in \mathrm{Lim}(\lambda^+)$ is called a good point *for $\vec{f}$ if there exist an unbounded $b \subseteq \alpha$ of order-type $\mathrm{cf}(\alpha)$ and $\nu < \sup(A)$ such that $\langle f_\beta \mid \beta \in b \rangle$ is $<_\nu$-increasing.*

(2) $\alpha \in \mathrm{Lim}(\lambda^+)$ is called a better point *for $\vec{f}$ if there exist a club $c \subseteq \alpha$ of order-type $\mathrm{cf}(\alpha)$ and $\sigma : c \to \sup(A)$ such that $f_\beta <_{\max(\sigma(\beta),\sigma(\gamma))} f_\gamma$ for each $\beta, \gamma \in c$ with $\beta < \gamma$.*

In general better points may not be good. But note that all better points of cofinality $> \mathrm{cf}(\sup(A))$ are good.

Definition 2.7. *Let λ be a singular cardinal and A be a set of regular cardinals with $\sup(A) = \lambda$. Suppose that $\vec{f} = \langle f_\alpha \mid \alpha < \lambda^+ \rangle$ is a scale in $\langle \Pi A, <^* \rangle$.*

(1) $\vec{f}$ is called a good scale *if every $\alpha \in E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ is a good point for $\vec{f}$.*

(2) $\vec{f}$ is called a better scale *if every $\alpha \in E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ is a better point for $\vec{f}$.*

2.2 Exact upper bound

Here we present basic facts on the existence of the exact upper bound, which are building blocks of PCF theory.

First we give a condition for a $<^*$-increasing sequence $\vec{f}$ in ${}^A\mathrm{On}$ which is equivalent to that $\vec{f}$ has the exact upper bound with respect to $<^*$:

Lemma 2.8. *Let λ be a singular cardinal and A be a set of regular cardinals with $|A| < \lambda = \sup(A)$. Moreover let μ be a regular cardinal such that $|A| < \mu < \lambda$. Then the following are equivalent for a $<^*$-increasing sequence $\vec{f}$ in ${}^A\mathrm{On}$ of length λ^+:*

(1) $\vec{f}$ has the exact upper bound g with respect to $<^*$ such that $\mathrm{cf}(g(\kappa)) \geq \mu$ for all $\kappa \in A$.

(2) There are stationary many good points for $\vec{f}$ in $E^{\lambda^+}_{\mu}$.

It is not hard to prove that (1) implies (2):

Proof of Lem.2.8 ((1) implies (2)). We may assume that $\mu < \min(A)$. Let $\vec{f} = \langle f_\alpha \mid \alpha < \lambda^+ \rangle$ be a $<^*$-increasing sequence in ${}^A\mathrm{On}$, and suppose that $\vec{f}$ has the exact upper bound g with respect to $<^*$ such that $\mathrm{cf}(g(\kappa)) \geq \mu$ for all $\kappa \in A$. Suppose also that C is a club subset of λ^+. We find a good point for $\vec{f}$ in $C \cap E^{\lambda^+}_\mu$.

Let θ be a sufficiently large regular cardinal. Take a sequence $\langle M_\xi \mid \xi < \mu \rangle$ in $[\mathcal{H}_\theta]^{<\mu}$ such that

- $M_\xi \subseteq M_{\xi+1}$, and $M_\xi \in M_{\xi+1}$,
- $A \subseteq M_\xi \prec \langle \mathcal{H}_\theta, \in, \lambda, \mu, A, \vec{f}, g, C \rangle$,

for each $\xi < \mu$. Let $M := \bigcup_{\xi<\mu} M_\xi$ and $\alpha := \sup(M \cap \lambda^+)$. Clearly $\alpha \in C \cap E^{\lambda^+}_\mu$. Thus it suffices to show that α is good for $\vec{f}$.

For each $\xi < \mu$ let $g_\xi \in {}^A\mathrm{On}$ be such that

$$g_\xi(\kappa) = \sup(M_\xi \cap g(\kappa))$$

for each $\kappa \in A$. Then note the following:

- $g_\xi < g$ and $g_\xi \in M_{\xi+1}$ for each $\xi < \mu$.
- $\langle g_\xi \mid \xi < \mu \rangle$ is $\leq$-increasing.
- $f_\beta <^* g_\xi$ for any $\beta < \sup(M \cap \lambda^+)$.

For the last one note that if $\beta \in M_\xi \cap \lambda^+$, then $f_\beta <^* g_\xi$ because $f_\beta[A] \subseteq M_\xi$, and $f_\beta <^* g$.

For each $\xi < \mu$ we can take $\beta_\xi \in M_{\xi+1} \cap \lambda^+$ such that $g_\xi <^* f_{\beta_\xi}$ because g is the exact upper bound of $\vec{f}$. Note that $\sup(M_\xi \cap \lambda^+) < \beta_\xi$ for each $\xi < \mu$. Thus $\langle \beta_\xi \mid \xi < \mu \rangle$ is increasing cofinal in α.

Because $\mathrm{cf}(\lambda) < \mu$, we can take $\nu < \lambda$ such that the set

$$b' \ := \ \{\xi < \mu \mid g_\xi <_\nu f_{\beta_\xi} <_\nu g_{\xi+1}\}$$

is unbounded in μ. Note that if $\xi, \eta \in b'$, and $\xi < \eta$, then $f_{\beta_\xi} <_\nu f_{\beta_\eta}$ because $f_{\beta_\xi} <_\nu g_{\xi+1} \leq g_\eta <_\nu f_{\beta_\eta}$. Thus $\langle f_{\beta_\xi} \mid \xi \in b' \rangle$ is $<_\nu$-increasing. Then $b := \{\beta_\xi \mid \xi \in b'\}$ witnesses that α is good for $\vec{f}$. □

To prove that (1) implies (2), we need preliminaries:

Notation 2.9. *Let A be a set of regular cardinals. Suppose that $f \in {}^A\mathrm{On}$ and that $\mathfrak{g}$ is a function on A such that $\mathfrak{g}(\kappa)$ is a set of ordinals for each $\kappa \in A$. Then let $\mathrm{proj}(f, \mathfrak{g})$ be the function on A such that*

$$\mathrm{proj}(f,\mathfrak{g})(\kappa) \ = \ \begin{cases} \min(\mathfrak{g}(\kappa) \setminus f(\kappa)) & \cdots \quad \textit{if } \mathfrak{g}(\kappa) \setminus f(\kappa) \neq \emptyset \\ 0 & \cdots \quad \textit{if } \mathfrak{g}(\kappa) \setminus f(\kappa) = \emptyset \end{cases}$$

for each $\kappa \in A$.

Lemma 2.10. *Let λ, A and μ be as in Lem.2.8. Suppose that $\vec{f} = \langle f_\alpha \mid \alpha < \lambda^+ \rangle$ is a $<^*$-increasing sequence in ${}^A\mathrm{On}$ and that there are stationary many good points for $\vec{f}$ in $E^{\lambda^+}_\mu$. Moreover let $\mathfrak{g}$ be a function from A to $[\mathrm{On}]^{<\mu}$. Then there exists $\alpha < \lambda^+$ such that $\mathrm{proj}(f_\alpha, \mathfrak{g}) =^* \mathrm{proj}(f_\beta, \mathfrak{g})$ for all $\beta \geq \alpha$.*

Proof. For the contradiction assume that there are no $\alpha < \lambda^+$ as in Lem.2.10.

Note that $\langle \mathrm{proj}(f_\beta, \mathfrak{g}) \mid \beta < \lambda^+ \rangle$ is $\leq^*$-increasing. Then we can take an unbounded $E \subseteq \lambda^+$ such that $\mathrm{proj}(f_\beta, \mathfrak{g}) \leq^* \mathrm{proj}(f_\gamma, \mathfrak{g})$ but $\mathrm{proj}(f_\beta, \mathfrak{g}) \not\geq^* \mathrm{proj}(f_\gamma, \mathfrak{g})$ for each $\beta, \gamma \in E$ with $\beta < \gamma$.

Take a good point $\alpha \in E^{\lambda^+}_\mu$ for $\vec{f}$ such that $E \cap \alpha$ is unbounded in α, and suppose that $b \subseteq \alpha$ and $\nu < \lambda$ witness goodness of α. By shrinking b if necessary, we may assume that $\mathrm{proj}(f_\beta, \mathfrak{g}) \not\geq^* \mathrm{proj}(f_\gamma, \mathfrak{g})$ for each $\beta, \gamma \in b$ with $\beta < \gamma$.

For each $\beta \in b$ take $\kappa_\beta \in A \setminus \nu$ with

$$\mathrm{proj}(f_\beta, \mathfrak{g})(\kappa_\beta) \ < \ \mathrm{proj}(f_{\min(b \setminus \beta+1)}, \mathfrak{g})(\kappa_\beta) \ .$$

Then we can take $\kappa \in A \setminus \nu$ such that $\kappa_\beta = \kappa$ for cofinally many $\beta \in b$ because $|A| < \mu = \mathrm{o.t.}(b)$. Note that $\langle \mathrm{proj}(f_\beta, \mathfrak{g})(\kappa) \mid \beta \in b \rangle$ is $\leq$-increasing because $\langle f_\beta \mid \beta \in b \rangle$ is $<_\nu$-increasing. Moreover it is not eventually constant by the choice of κ. This contradicts that $|\mathfrak{g}(\kappa)| < \mu$. □

Lemma 2.11. *Let λ, A and μ be as in Lem.2.8. Suppose that $\vec{f} = \langle f_\alpha \mid \alpha < \lambda^+ \rangle$ is a $<^*$-increasing sequence in ${}^A\mathrm{On}$ and that there are stationary many good points for $\vec{f}$ in $E^{\lambda^+}_\mu$. Then $\vec{f}$ has the least upper bound with respect to $\leq^*$.*

Proof. For the contradiction assume that $\vec{f}$ has the least upper bound with respect to $\leq^*$.

By induction on $\xi < \mu$ take $g_\xi \in {}^A\mathrm{On}$, $\mathfrak{g}_\xi \in {}^A([\mathrm{On}]^{<\mu})$ and $\alpha_\xi < \lambda^+$ as follows: Let $\mathfrak{g}_0$ be the constant function on A with its value $\emptyset$, and let $\alpha_0 := 0$. Moreover let g_0 be an arbitrary upper bound of $\vec{f}$ with respect to $\leq^*$. Assume that $0 < \xi < \mu$ and that g_η and α_η have been defined for all $\eta < \xi$. First let $\mathfrak{g}_\xi$ be the function on A such that $\mathfrak{g}_\xi(\kappa) = \{ g_\eta(\kappa) \mid \eta < \xi \}$ for each $\kappa \in A$. Then by Lem.2.10 we can take $\alpha_\xi < \lambda^+$ such that $\mathrm{proj}(f_\beta, \mathfrak{g}_\xi) =^* \mathrm{proj}(f_{\alpha_\xi}, \mathfrak{g}_\xi)$ for every $\beta \geq \alpha_\xi$. Note that $\mathrm{proj}(f_{\alpha_\xi}, \mathfrak{g}_\xi)$ is an upper bound of $\vec{f}$ with respect to $\leq^*$. Then by our assumption on the non-existence of the least upper bound, let g_ξ be an upper bound of $\vec{f}$ with respect to $\leq^*$ such that $\mathrm{proj}(f_{\alpha_\xi}, \mathfrak{g}_\xi) \not\leq^* g_\xi$.

Take $\alpha < \lambda^+$ such that $\alpha \geq \alpha_\xi$ for every $\xi < \mu$, and let $h_\xi := \mathrm{proj}(f_\alpha, \mathfrak{g}_\xi)$ for each $\xi < \mu$. Note that $h_{\xi+1} \leq g_\xi$ and that $h_\xi \not\leq^* g_\xi$. Hence $h_\xi \not\leq^* h_{\xi+1}$ for each $\xi < \mu$. Thus for each $\xi < \mu$ there exists $\kappa_\xi \in A$ with $h_{\xi+1}(\kappa_\xi) < h_\xi(\kappa_\xi)$. Then we can take $\kappa \in A$ such that $\kappa_\xi = \kappa$ for cofinally many $\xi < \mu$ because

$|A| < \mu$. Here note that $\langle h_\xi(\kappa) \mid \xi < \mu \rangle$ is $\leq$-decreasing. Thus $\langle h_\xi(\kappa) \mid \xi < \mu \rangle$ includes an infinite $<$-decreasing subsequence. This is a contradiction. □

Now we prove Lem.2.8:

Proof of Lem.2.8. By Lem.2.11 let $g \in {}^A\mathrm{On}$ be the least upper bound of $\vec{f}$ with respect to $\leq^*$. We prove that g is the exact upper bound and that $\mathrm{cf}(g(\kappa)) \geq \mu$ for all sufficiently large $\kappa \in A$.

First we prove that g is the exact upper bound. To prove this, take an arbitrary $h \in {}^A\mathrm{On}$ such that $h <^* g$. Let $\mathfrak{g}$ be the function on A such that $\mathfrak{g}(\kappa) = \{g(\kappa), h(\kappa)\}$ for each $\kappa \in A$. Then by Lem.2.10 we can take $\alpha < \lambda^+$ such that $\mathrm{proj}(f_\beta, \mathfrak{g}) =^* \mathrm{proj}(f_\alpha, \mathfrak{g})$ for every $\beta \geq \alpha$. Then $\mathrm{proj}(f_\alpha, \mathfrak{g})$ is an upper bound of $\vec{f}$ with respect to $\leq^*$. Hence $g \leq^* \mathrm{proj}(f_\alpha, \mathfrak{g})$. Then it is easy to see that $h <^* f_\alpha$.

Next we prove that $\mathrm{cf}(g(\kappa)) \geq \mu$ for all sufficiently large $\kappa \in A$. For the contradiction assume not, i.e. assume that $B := \{\kappa \in A \mid \mathrm{cf}(g(\kappa)) < \mu\}$ is unbounded in λ. For each $\kappa \in B$ take a cofinal $b_\kappa \subseteq g(\kappa)$ of order-type $< \mu$, and let $\mathfrak{g}$ be a function on A such that

$$\mathfrak{g}(\kappa) \;=\; \begin{cases} b_\kappa \cup \{g(\kappa)\} & \cdots \quad \text{if } \kappa \in B \\ \{g(\kappa)\} & \cdots \quad \text{if } \kappa \in A \setminus B \end{cases} .$$

Then by the same argument as above we can take $\alpha < \lambda^+$ such that $g \leq^* \mathrm{proj}(f_\alpha, \mathfrak{g})$. Then $f_\alpha =^* g$ by the construction of $\mathfrak{g}$. This contradicts that $f_\alpha <^* f_{\alpha+1} \leq^* g$. □

Using Lem.2.8 and a well-known fact on the ideal $I[\nu]$, we can prove the following:

Lemma 2.12. *Let λ be a singular cardinal, and let A be a set of regular cardinals with $|A| < \lambda = \sup(A)$. Then for any sequence $\vec{h} = \langle h_\alpha \mid \alpha < \mu^+ \rangle$ in ΠA there exists a $<^*$-increasing sequence $\vec{f} = \langle f_\alpha \mid \alpha < \lambda^+ \rangle$ with the following properties:*

(i) $\vec{f}$ has the exact upper bound g with respect to $<^*$ such that $\langle \mathrm{cf}(g(\kappa)) \mid \kappa \in A \rangle$ converges to λ, i.e. for any $\mu < \lambda$ the set $\{\kappa \in A \mid \mathrm{cf}(g(\kappa)) \leq \mu\}$ is bounded in λ.

(ii) $h_\alpha < f_\alpha$ for every $\alpha < \mu^+$.

Recall the ideal $I[\nu]$ and a well-known fact on it, due to Shelah [2], before proving Lem.2.12:

Definition 2.13. *For a regular cardinal $\nu \geq \omega_2$ let $I[\nu]$ be the set of all $E \subseteq \nu$ with the following property:*

There exists a sequence $\langle a_\alpha \mid \alpha < \nu\rangle$ of bounded subsets of ν and a club $C \subseteq \mathrm{Lim}(\nu)$ such that

(i) $\mathrm{o.t.}(a_\alpha) = \mathrm{cf}(\alpha)$,

(ii) $\{a_\alpha \cap \gamma \mid \gamma < \alpha\} \subseteq \{a_\beta \mid \beta < \alpha\}$,

for all $\alpha \in E \cap C$.

Fact 2.14 (Shelah [2])**.** *Let ν be a regular cardinal $\geq \omega_3$. Then there exists $E \in I[\nu]$ such that $E \cap E^\nu_\mu$ is stationary for every regular μ with $\mu^{++} < \nu$.*

Proof of Lem.2.12. Take an arbitrary sequence $\langle h_\alpha \mid \alpha < \lambda^+\rangle$ in ΠA. We construct $\vec{f}$ satisfying the properties (i) and (ii) in Lem.2.12.

By Fact 2.14 take $E \in I[\lambda^+]$ such that $E \cap E^{\lambda^+}_\mu$ is stationary for all regular $\mu < \lambda$. Suppose that $\langle a_\alpha \mid \alpha < \lambda^+\rangle$ and $C \subseteq \mathrm{Lim}(\lambda^+)$ witness that $E \in I[\lambda^+]$. We may assume that $\langle a_\alpha \mid \alpha < \lambda^+\rangle$ is one to one and that $\mathrm{o.t.}(a_\alpha) < \lambda$ and $a_\alpha \subseteq \alpha$ for every $\alpha < \lambda^+$. For each $\alpha < \lambda^+$ let b_α be the set of all $\beta < \alpha$ such that a_β is an initial segment of a_α. Note the following:

- Each b_α is a subset of α of order-type $< \lambda$.
- If $\beta \in b_\alpha$, then $b_\beta = b_\alpha \cap \beta$.
- If $\alpha \in E \cap C$, then b_α is an unbounded subset of α of order-type $\mathrm{cf}(\alpha)$.

By induction on $\alpha < \lambda^+$ take $f_\alpha \in \Pi A$ so that

- $f_\beta <^* f_\alpha$ for all $\beta < \alpha$,
- $h_\alpha <^* f_\alpha$,
- $\sup\{f_\beta \mid \beta \in b_\alpha\} <_{\mathrm{o.t.}(b_\alpha)} f_\alpha$.

We can take such f_α because $\langle \Pi A, <^*\rangle$ is $\leq \lambda$-directed, and $\mathrm{o.t.}(b_\alpha) < \lambda$.

Clearly $\vec{f}$ satisfies the property (ii) in Lem.2.12. Note that if $\alpha \in E \cap C$, then $\langle f_\beta \mid \beta \in b_\alpha\rangle$ is $<_{\mathrm{o.t.}(b_\alpha)}$-increasing. Hence each $\alpha \in E \cap C$ is a good point for $\vec{f}$. Then $\vec{f}$ satisfies (i) in Lem.2.12 by the choice of E and Lem.2.8. □

2.3 Maximal ideals and bounded ideal

Here we review the relationship between $\langle \Pi A, <_I\rangle$ for maximal ideals I and $\langle \Pi A, <^*\rangle$. More precisely we prove Lem.2.15 and 2.16 below:

Lemma 2.15. *Let λ be a singular cardinal and A be a set of regular cardinals with $|A| < \lambda = \sup(A)$. Assume that* $\mathrm{tcf}(\Pi A, <_I) = \lambda^+$ *for every ideal I over A including the bounded ideal. Then* $\mathrm{tcf}(\Pi A, <^*) = \lambda^+$.

Lemma 2.16. *Let λ is a singular cardinal and A be a set of regular cardinals with $|A| < \lambda = \sup(A)$. Assume that there exists a maximal ideal I over A which includes the bounded ideal and such that $\mathrm{tcf}(\Pi A, <_I) > \lambda$. Then there exists an unbounded $B \subseteq A$ such that $\langle \Pi B, <^* \rangle$ is $\leq \lambda^+$-directed.*

We use the following lemma. A sequence $\vec{f}$ in the following lemma is called a *universal sequence* in ΠA of length λ^+:

Lemma 2.17. *Suppose that λ is a singular cardinal and that A is a set of regular cardinals with $|A| < \lambda = \sup(A)$. Then there exists a $<^*$-increasing sequence $\vec{f}$ of length λ^+ with the following properties:*

(i) $\vec{f}$ has the exact upper bound with respect to $<^*$.

(ii) $\vec{f}$ is $<_I$-cofinal in ΠA for any maximal ideal I over A which includes the bounded ideal and such that $\mathrm{tcf}(\Pi A, <_I) = \lambda^+$.

Proof. By reducing A if necessary, we may assume that $\min(A) > |A|^+$. For the contradiction assume that there are no such $\vec{f}$. Note that this assumption together with Lem.2.12 implies the existence of a maximal ideal I over A which includes the bounded ideal and such that $\mathrm{tcf}(\Pi A, <_I) = \lambda^+$.

By induction on $\xi < |A|^+$ we define a $<^*$-increasing sequence $\vec{f_\xi} = \langle f_{\xi,\alpha} \mid \alpha < \lambda^+ \rangle$ in ΠA and a maximal ideal I_ξ over A. Assume that $\xi < |A|^+$ and that $\vec{f_\eta}$ and I_η have been defined for every $\eta < \xi$. First take a maximal ideal I_ξ over A such that

- I_ξ includes the bounded ideal,
- $\mathrm{tcf}(\Pi A, <_{I_\xi}) = \lambda^+$,
- if ξ is a successor ordinal, then $\vec{f}_{\xi-1}$ is not $<_{I_\xi}$-cofinal in ΠA.

Next take a $<^*$-increasing sequence $\vec{f_\xi} = \langle f_{\xi,\alpha} \mid \alpha < \lambda^+ \rangle$ in ΠA such that

- $\vec{f_\xi}$ has the exact upper bound with respect to $<^*$,
- $\vec{f_\xi}$ is $<_{I_\xi}$-cofinal in ΠA,
- $\sup\{f_{\eta,\alpha} \mid \eta < \xi\} \leq f_{\xi,\alpha}$ for all $\alpha < \lambda^+$,
- if ξ is a successor ordinal, then $f_{\xi-1,\alpha} <_{I_\xi} f_{\xi,0}$ for all $\alpha < \lambda^+$.

We can take such $\vec{f_\xi}$ by Lem.2.12 and the fact that $\xi < |A|^+ < \min(A)$.

Let $f := \sup\{f_{\xi,0} \mid \xi < |A|^+\} \in \Pi A$. For each $\xi < |A|^+$ take $\alpha_\xi < \lambda^+$ such that $f <_{I_\xi} f_{\xi,\alpha_\xi}$. Moreover take $\beta < \lambda^+$ such that $\beta > \alpha_\xi$ for all $\xi < |A|^+$. Let $A_\xi := \{\kappa \in A \mid f(\kappa) < f_{\xi,\beta}(\kappa)\}$ for each $\xi < |A|^+$.

Note that $\langle A_\xi \mid \xi < |A|^+\rangle$ is $\subseteq$-increasing because $\langle f_{\xi,\beta} \mid \xi < |A|^+\rangle$ is $\leq$-increasing. Moreover

$$f_{\xi,\beta} \ <_{I_{\xi+1}} \ f_{\xi+1,0} \ \leq \ f \ <_{I_{\xi+1}} \ f_{\xi+1,\beta}$$

for each $\xi < |A|^+$. Hence $A_\xi \in I_{\xi+1}$, and $A_{\xi+1} \notin I_{\xi+1}$. So $\langle A_\xi \mid \xi < |A|^+\rangle$ is a $\subset$-increasing sequence of subsets of A. This is a contradiction. □

Using Lem.2.17 we can prove Lem.2.15 and 2.16 easily:

Proof of Lem.2.15. Let $\vec{f} = \langle f_\alpha \mid \alpha < \lambda^+\rangle$ be a universal sequence in ΠA, and let f be the exact upper bound of $\vec{f}$ with respect to $<^*$. We may assume that $f(\kappa) \leq \kappa$ for all $\kappa \in A$.

Assume that $B := \{\kappa \in A \mid f(\kappa) < \kappa\}$ is unbounded in A. Then we can take a maximal ideal I over A including the bounded ideal and containing $A \setminus B$. Then $\mathrm{tcf}(\Pi A, <_I) = \lambda^+$ by the assumption in Lem.2.15, but $\vec{f}$ is not $<_I$-cofinal in ΠA. This contradicts that $\vec{f}$ is a universal sequence. So B is bounded in A, and this implies that $\vec{f}$ is a scale in $\langle \Pi A, <^*\rangle$. Therefore $\mathrm{tcf}(\Pi A, <^*) = \lambda^+$. □

Proof of Lem.2.16. Let $\vec{f} = \langle f_\alpha \mid \alpha < \lambda^+\rangle$ be a universal sequence in ΠA, and let f be the exact upper bound of $\vec{f}$ with respect to $<^*$. We may assume that $f(\kappa) \leq \kappa$ for all $\kappa \in A$. We show that $B := \{\kappa \in A \mid f(\kappa) < \kappa\}$ witnesses Lem.2.16.

First note that if B is bounded in A, then $\vec{f}$ is a scale in $\langle \Pi A, <^*\rangle$, and so is in $\langle \Pi A, <_I\rangle$ for all maximal ideal I over A including the bounded ideal. This contradicts the assumption in Lem.2.16. Thus B is unbounded in A.

To show that B is $\leq \lambda^+$-unbounded, take an arbitrary $\mathcal{G} \subseteq \Pi B$ of cardinality $\leq \lambda^+$. By Lem.2.12 we can take a $<^*$-increasing sequence $\langle h_\alpha \mid \alpha < \lambda^+\rangle$ which has the exact upper bound h with respect to $<^*$ and such that for any $g \in \mathcal{G}$ there exists $\alpha < \lambda^+$ with $g \leq h_\alpha$. We may assume that $h(\kappa) \leq \kappa$ for all $\kappa \in B$. All we have to show is the set $C := \{\kappa \in B \mid h(\kappa) = \kappa\}$ is bounded in B. (Then h yields an upper bound of $\mathcal{G}$ in $\langle \Pi B, <^*\rangle$.)

Assume not. Take a maximal ideal I over A including the bounded ideal and containing $A \setminus C$. Note that $\langle h_\alpha \restriction C \mid \alpha < \lambda^+\rangle$ is a scale in $\langle \Pi C, <^*\rangle$. Hence it yields a scale in $\langle \Pi A, <_I\rangle$ of length λ^+. That is, $\mathrm{tcf}(\Pi A, <_I) = \lambda^+$. But f is an upper bound of $\vec{f}$ in $\langle \Pi A, <_I\rangle$ because $A \setminus B \in I$. This contradicts that $\vec{f}$ is a universal sequence. □

We end this section with corollaries of Lem.2.15 and 2.16 on the pseudo power:

Corollary 2.18. *The following are equivalent for a singular cardinal λ:*

(1) $\mathrm{pp}(\lambda) = \lambda^+$.

(2) $\mathrm{tcf}(\Pi A, <_I) = \lambda^+$ *for every set* A *of regular cardinals with* $|A| < \lambda = \sup(A)$ *and every maximal ideal* I *over* A *including the bounded ideal.*

Proof. Clearly (2) implies (1). We prove the reverse implication.

Assume that (2) fails. Then there are a set A of regular cardinals with $|A| < \lambda = \sup(A)$ and a maximal ideal I over A including the bounded ideal such that $\mathrm{tcf}(\Pi A, <_I) > \lambda^+$. Then by Lem.2.16 we can take an unbounded $B \subseteq A$ such that $\langle \Pi B, <^* \rangle$ is $\leq \lambda^+$-directed. By shrinking if necessary, we may assume that $\mathrm{o.t.}(B) = \mathrm{cf}(\lambda)$. Because $\langle \Pi B, <^* \rangle$ is $\leq \lambda^+$-directed, $\mathrm{tcf}(\Pi B, <_J) > \lambda^+$ for any maximal ideal J over B including the bounded ideal. Therefore $\mathrm{pp}(\lambda) > \lambda^+$, i.e. (1) fails. □

Corollary 2.19. *Assume that* λ *is a singular cardinal with* $\mathrm{pp}(\lambda) = \lambda^+$. *Then* $\mathrm{tcf}(\Pi A, <^*) = \lambda^+$ *for any set* A *of regular cardinals with* $|A| < \lambda = \sup(A)$.

Proof. This is clear from Lem.2.15 and Cor.2.18. □

3 Silver's theorem for SCH^+

Here we prove Thm.1.1:

Theorem 1.1. *Assume that* SCH^+ *fails, and let* λ *be the least singular cardinal at which* SCH^+ *fails. Then* $\mathrm{cf}(\lambda) = \omega$.

We use the following lemmata:

Lemma 3.1. *Let* λ *be a singular cardinal of uncountable cofinality. Then there exists a club* $C \subseteq \lambda$ *such that* $\langle \Pi C^+, <^* \rangle$ *has a scale of length* λ^+, *where* $C^+ := \{\nu^+ \mid \nu \in C\}$.

Proof. First take a club $B \subseteq \lambda$ which consists of singular cardinals and such that $\mathrm{o.t.}(B) = \mathrm{cf}(\lambda) < \min(B)$. Then by Lem.2.12 we can take a $<^*$-increasing sequence $\vec{f} = \langle f_\alpha \mid \alpha < \lambda^+ \rangle$ in ΠB^+ and the exact upper bound g of $\vec{f}$ with respect to $<^*$ such that $\langle \mathrm{cf}(g(\nu^+)) \mid \nu \in B \rangle$ converges to λ. We may assume that $g(\nu^+) \leq \nu^+$ for each $\nu \in B$.

Note that if $g(\nu^+) < \nu^+$, then $\mathrm{cf}(g(\nu^+)) < \nu$. Thus if there are stationary many $\nu \in B$ with $g(\nu^+) < \nu^+$, then, by Fodor's lemma, we can take $\mu < \lambda$ such that $\mathrm{cf}(g(\nu^+)) < \mu$ for stationary many $\nu \in B$. This contradicts that $\langle \mathrm{cf}(g(\nu^+)) \mid \nu \in B \rangle$ converges to λ. Therefore there are club many $\nu \in B$ with $g(\nu^+) = \nu^+$. Let C be a club subset of B consisting of ν with $g(\nu^+) = \nu^+$. Then $\langle f_\alpha \restriction C^+ \mid \alpha < \mu^+ \rangle$ is a scale in $\langle \Pi C^+, <^* \rangle$. □

Lemma 3.2. *Assume that λ is a singular cardinal and that* $\mathrm{cf}([\nu]^{\mathrm{cf}(\nu)}, \subseteq) = \nu^+$ *for all singular cardinals $\nu < \lambda$. Then* $\mathrm{cf}([\nu]^\mu, \subseteq) = \nu$ *for all regular cardinals μ, ν with $\mu < \nu < \lambda$.*

Proof. Take an arbitrary regular cardinal $\mu < \lambda$. We prove the lemma by induction on ν.

If $\nu = \mu^+$, then $\mathrm{cf}([\nu]^\mu, \subseteq) = \nu$ because ν is $\subseteq$-cofinal in $[\nu]^\mu$. If ν is the successor cardinal of a regular cardinal $\nu' > \mu$, then

$$\mathrm{cf}([\nu]^\mu, \subseteq) \;=\; \mathrm{cf}([\nu]^{\nu'}, \subseteq) \cdot \mathrm{cf}([\nu']^\mu, \subseteq) \;=\; \nu \cdot \nu' \;=\; \nu \,.$$

If ν is a limit cardinal, then

$$\mathrm{cf}([\nu]^\mu, \subseteq) \;=\; \sup\{\mathrm{cf}([\nu']^\mu, \subseteq) \mid \nu' \text{ is a regular cardinal} < \nu\} \;=\; \nu \,.$$

Suppose that ν is the successor cardinal of a singular cardinal $\nu' > \mu$. Then

$$\mathrm{cf}([\nu]^\mu, \subseteq) \;=\; \mathrm{cf}([\nu]^{\nu'}, \subseteq) \cdot \mathrm{cf}([\nu']^\mu, \subseteq) \;=\; \nu \cdot \mathrm{cf}([\nu']^\mu, \subseteq) \,.$$

Moreover it is easy to see that

$$\mathrm{cf}([\nu']^\mu, \subseteq) \;\leq\; \mathrm{cf}([\nu']^{\mathrm{cf}(\nu')}, \subseteq) \;=\; \nu$$

because $\mathrm{cf}([\nu'']^\mu, \subseteq) = \nu''$ for all regular $\nu'' < \nu$. So $\mathrm{cf}([\nu]^\mu, \subseteq) = \nu$. ☐

Now we prove Thm.1.1:

Proof of Thm.1.1. Assume that λ is a singular cardinal of uncountable cofinality and that $\mathrm{cf}([\nu]^{\mathrm{cf}(\nu)}, \subseteq) = \nu^+$ for all singular cardinals $\nu < \lambda$. We show that $\mathrm{cf}([\lambda]^{\mathrm{cf}(\lambda)}, \subseteq) = \lambda^+$. It suffices to find a $\subseteq$-cofinal $X \subseteq [\lambda]^{\leq \mathrm{cf}(\lambda)}$ of cardinality λ^+.

By Lem.3.1 take a club $C \subseteq \lambda$ and a scale $\vec{f} = \langle f_\alpha \mid \alpha < \lambda^+ \rangle$ in $\langle \Pi C^+, <^* \rangle$. We may assume that $\mathrm{o.t.}(C) = \mathrm{cf}(\lambda)$ and that C consists of singular cardinals $> \mathrm{cf}(\lambda)$. For each μ with $\mathrm{cf}(\lambda) < \mu < \lambda$ take a $\subseteq$-cofinal subset $\{y_{\mu,\gamma} \mid \gamma < \mu\}$ of $[\mu]^{\mathrm{cf}(\lambda)}$. Moreover for each $\nu \in C$ and $\gamma < \nu^+$ take an injection $\sigma_{\nu,\gamma} : \gamma \to \nu$.

Now for each $\alpha < \lambda^+$, each regular μ with $\mathrm{cf}(\lambda) < \mu < \lambda$ and each $\delta < \mu$ let

$$x_{\alpha,\mu,\delta} \;:=\; \bigcup \{y_{\nu^+,\gamma} \mid \nu \in C \,\wedge\, \gamma < f_\alpha(\nu^+) \,\wedge\, \sigma_{\nu, f_\alpha(\nu^+)}(\gamma) \in y_{\mu,\delta}\} \,.$$

Note that $x_{\alpha,\mu,\delta} \in [\lambda]^{\leq \mathrm{cf}(\lambda)}$. Let X be the collection of all such $x_{\alpha,\mu,\delta}$'s. Then $|X| = \lambda^+$ clearly.

We prove that X is $\subseteq$-cofinal. Take an arbitrary $x \in [\lambda]^{\mathrm{cf}(\lambda)}$. Let $f \in \Pi C^+$ be such that $x \cap \nu^+ \subseteq y_{\nu^+, f(\nu^+)}$ for each $\nu \in C$. Then we can take $\alpha < \lambda^+$

such that $f <^* f_\alpha$. Then by Fodor's lemma we can take a regular μ such that $\mathrm{cf}(\lambda) < \mu < \lambda$ and such that the set

$$B \;:=\; \{\nu \in C \mid f(\nu^+) < f_\alpha(\nu^+) \,\wedge\, \sigma_{\nu, f_\alpha(\nu^+)}(f(\nu^+)) < \mu\}$$

is stationary in λ. Then we can take $\delta < \mu$ such that

$$\{\sigma_{\nu, f_\alpha(\nu^+)}(f(\nu^+)) \mid \nu \in B\} \;\subseteq\; y_{\mu,\delta} \,.$$

Note that $x \cap \nu^+ \subseteq x_{\alpha,\mu,\delta}$ for each $\nu \in B$. Then $x \subseteq x_{\alpha,\mu,\delta}$ because B is unbounded in λ. □

4 Pseudo power

In this section we prove Thm.1.2:

Theorem 1.2. *Assume that* SCH^+ *fails, and let* λ *be the least singular cardinal at which* SCH^+ *fails. Then* $\mathrm{pp}(\lambda) > \lambda^+$*.*

For this we need preliminaries. First we prove a lemma on the relationship between $\mathrm{cf}([\lambda]^{\mathrm{cf}(\lambda)}, \subseteq)$ and the product of all regular cardinals below λ:

Lemma 4.1. *Suppose that* λ *is a singular cardinal such that* $\mathrm{cf}([\nu]^{\mathrm{cf}(\lambda)}, \subseteq) < \lambda$ *for all* $\nu < \lambda$*. Let* R *be the set of all regular cardinals* $< \lambda$*. Then the following are equivalent:*

(1) $\mathrm{cf}([\lambda]^{\mathrm{cf}(\lambda)}, \subseteq) = \lambda^+$*.*

(2) There exists $\mathcal{F} \subseteq \Pi R$ *of cardinality* λ^+ *such that for any* $A \subseteq R$ *with* $\sup(A) = \lambda$ *and* $|A| = \mathrm{cf}(\lambda)$ *and for any* $g \in \Pi A$ *there exists* $f \in \mathcal{F}$ *with* $g <^* f \restriction A$*.*

We can easily prove that (1) implies (2) by a similar argument as in the proof of Prop.2.5. The main part of Lem.4.1 is that (2) implies (1). The following lemma is a core of Lem.4.1:

Lemma 4.2. *Let* λ *be a singular cardinal and* R *be the set of all regular cardinals below* λ*. Assume (2) in Lem.4.1. Moreover let* θ *be a sufficiently large regular cardinal,* Δ *be a well-ordering of* $\mathcal{H}_\theta$ *and* $\mathcal{M}$ *be the structure* $\langle \mathcal{H}_\theta, \in, \Delta, \lambda \rangle$*. Suppose that* $M, N \in [\mathcal{H}_\theta]^{\mathrm{cf}(\lambda)^+}$ *satisfies the following properties:*

(i) $M, N \prec \mathcal{M}$*.*

(ii) Both M *and* N *are internally approachable of length* $\mathrm{cf}(\lambda)^+$*.*

(iii) $\sup(M \cap \lambda^+) = \sup(N \cap \lambda^+)$.

Then there exists $\nu < \lambda$ *such that* $N \cap \lambda \subseteq \mathrm{Sk}^{\mathcal{M}}(M \cup \nu)$.

Proof. First we define $\nu < \lambda$ witnessing the lemma. For this we need preparations.

By (2) in Lem.4.1 and the $\leq \lambda$-directedness of $\langle \Pi R, <^* \rangle$ we can take a $<^*$-increasing sequence $\vec{f} = \langle f_\alpha \mid \alpha < \lambda^+ \rangle$ such that for any cofinal $A \subseteq R$ with $|A| = \mathrm{cf}(\lambda)$ and any $g \in \Pi A$ there exists $\alpha < \lambda^+$ with $g <^* f_\alpha \restriction A$. Let $\vec{f}$ is the Δ-least such sequence. Note that $\vec{f} \in M, N$.

Let $\langle N_\xi \mid \xi < \mathrm{cf}(\lambda)^+ \rangle$ be an internally approaching sequence to N. We may assume that $|N_\xi| = \mathrm{cf}(\lambda)$ for all $\xi < \mathrm{cf}(\lambda)^+$ and that $\sup(R \cap N_0) = \lambda$. For each $\xi < \mathrm{cf}(\lambda)^+$ let $g_\xi \in \Pi(R \cap N_\xi)$ be such that $g_\xi(\kappa) = \sup(N_\xi \cap \kappa)$ for each $\kappa \in R \cap N_\xi$ with $\kappa > \mathrm{cf}(\lambda)$.

Let $\alpha := \sup(M \cap \lambda^+) = \sup(N \cap \lambda^+)$. Note that both $M \cap \lambda^+$ and $N \cap \lambda^+$ are countably closed. Hence $M \cap N \cap \lambda^+$ is unbounded in α. Note also that $g_\xi \in N$ for each ξ. Then, by the choice of $\vec{f}$, for each $\xi < \mathrm{cf}(\lambda)^+$ we can take $\beta_\xi \in M \cap N \cap \lambda^+$ such that $g_\xi <^* f_{\beta_\xi} \restriction (R \cap N_\xi)$. Then we can take $\nu < \lambda$ such that $\nu > \mathrm{cf}(\lambda)$ and such that the set

$$b \ := \ \{\xi < \mathrm{cf}(\lambda)^+ \mid g_\xi <_\nu f_{\beta_\xi} \restriction (R \cap N_\xi)\}$$

is unbounded in $\mathrm{cf}(\lambda)^+$.

We prove that this ν witnesses the lemma. Let $\bar{M} := \mathrm{Sk}^{\mathcal{M}}(M \cup \nu)$. We must show that $N \cap \lambda \subseteq \bar{M}$. We claim the following:

Claim 1. $N \cap \bar{M} \cap \kappa$ *is unbounded in* $N \cap \kappa$ *for every* $\kappa \in R \cap N \cap \bar{M}$.

Proof of Claim 1. Fix $\kappa \in R \cap N \cap \bar{M}$. If $\kappa \leq \nu$, then the claim is clear because $\kappa \subseteq \bar{M}$. So suppose that $\kappa > \nu$. Note that $\kappa > \mathrm{cf}(\lambda)$.

First note that $\{f_{\beta_\xi}(\kappa) \mid \xi \in b\} \subseteq N \cap \bar{M} \cap \kappa$ because $\vec{f}, \kappa \in N \cap \bar{M}$, and $\beta_\xi \in N \cap \bar{M}$ for each $\xi \in H$. Hence it suffices to show that the set $\{f_{\beta_\xi}(\kappa) \mid \xi \in b\}$ is unbounded in $N \cap \kappa$.

For this note that if $\xi \in b$, and $\kappa \in N_\xi$, then $\sup(N_\xi \cap \kappa) = g_\xi(\kappa) < f_{\beta_\xi}(\kappa)$ because $\mathrm{cf}(\lambda), \nu < \kappa$. Moreover

$$N \cap \kappa \ = \ \bigcup \{N_\xi \cap \kappa \mid \xi \in b \ \wedge \ \kappa \in N_\xi\}$$

because b is unbounded in $\mathrm{cf}(\lambda)^+$. Therefore $\{f_{\beta_\xi}(\kappa) \mid \xi \in b\}$ is unbounded in $N \cap \kappa$. $\square_{\text{Claim1}}$

Claim 2. $N \cap \bar{M} \cap \delta$ *is unbounded in* $N \cap \delta$ *for every limit ordinal* $\delta \in N \cap \bar{M} \cap \lambda$.

Proof of Claim 2. Fix a limit ordinal $\delta \in N \cap \bar{M} \cap \lambda$. Then $\mathrm{cf}(\delta) \in R \cap N \cap \bar{M}$. Take an increasing continuous cofinal $\sigma : \mathrm{cf}(\delta) \to \delta$ in $N \cap \bar{M}$. Note that $\sup(N \cap \mathrm{cf}(\delta)) = \sup(N \cap \bar{M} \cap \mathrm{cf}(\delta))$ by the previous claim. Then by the elementarity of N and $\bar{M}$ it is easy to see that

$$\sup(N \cap \delta) = \sigma(\sup(N \cap \mathrm{cf}(\delta))) = \sigma(\sup(N \cap \bar{M} \cap \mathrm{cf}(\delta))) = \sup(N \cap \bar{M} \cap \delta)$$

if $\sup(N \cap \mathrm{cf}(\delta)) = \sup(N \cap \bar{M} \cap \mathrm{cf}(\delta)) < \mathrm{cf}(\delta)$. Otherwise, it is also easy to see that $\sup(N \cap \delta) = \sup(N \cap \bar{M} \cap \delta) = \delta$. $\square_{\mathrm{Claim2}}$

Using Claim 2, we can easily show that $N \cap \lambda \subseteq \bar{M}$: For the contradiction assume that $\gamma \in N \cap \lambda$ and that $\gamma \notin \bar{M}$. Let δ be $\min(N \cap \bar{M})$. Note that $\delta < \lambda$ because $N \cap \bar{M} \cap \lambda$ is unbounded in λ. Moreover δ is a limit ordinal $> \gamma$ by the elementarity of N and $\bar{M}$. Then we can take $\delta' > \gamma$ in $N \cap \bar{M} \cap \delta$ by Claim 2. This contradicts the choice of δ. $\square$

Lem.4.1 easily follows from Lem.4.2:

Proof of Lem.4.1. By a similar argument as in the proof of Prop.2.5 it is easily proved that (1) implies (2). We prove that (2) implies (1). Assume (1). We will find a $\subseteq$-cofinal $X \subseteq [\lambda]^{\mathrm{cf}(\lambda)}$ of cardinality λ^+.

Take a sufficiently large regular cardinal θ and a well-ordering Δ of $\mathcal{H}_\theta$. Let $\mathcal{M}$ be the structure $\langle \mathcal{H}_\theta, \in, \Delta, \lambda \rangle$. Moreover let

$$\begin{aligned} Z &:= \{M \in [\mathcal{H}_\theta]^{\mathrm{cf}(\lambda)^+} \mid M \prec \mathcal{M} \wedge M \text{ is i.a. of length } \mathrm{cf}(\lambda)^+\} , \\ E &:= \{\sup(M \cap \lambda^+) \mid M \in Z\} . \end{aligned}$$

For each $\alpha \in E$ choose $M_\alpha \in Z$ such that $\sup(M_\alpha \cap \lambda^+) = \alpha$. Moreover for each $\alpha \in E$ and $\nu < \lambda$ let $M_{\alpha,\nu}$ be $\mathrm{Sk}^{\mathcal{M}}(M_\alpha \cup \nu)$. Note that $|M_{\alpha,\nu}| < \lambda$. By the assumption of Lem.4.1 take a $\subseteq$-cofinal $X_{\alpha,\nu} \subseteq [M_{\alpha,\nu} \cap \lambda]^{\mathrm{cf}(\lambda)}$ of cardinality $< \lambda$ for each $\alpha \in E$ and $\nu < \lambda$. Then let $X := \bigcup\{X_{\alpha,\nu} \mid \alpha \in E \wedge \nu < \lambda\}$.

Clearly $X \subseteq [\lambda]^{\mathrm{cf}(\lambda)}$, and $|X| = \lambda^+$. Thus it suffices to show that X is $\subseteq$-cofinal in $[\lambda]^{\mathrm{cf}(\lambda)}$.

Take an arbitrary $y \in [\lambda]^{\mathrm{cf}(\lambda)}$. We find $x \in X$ with $x \supseteq y$. First we can take $N \in Z$ with $y \subseteq N$. Let $\alpha := \sup(N \cap \lambda^+) \in E$. By Lem.4.2 there exists $\nu < \lambda$ with $N \cap \lambda \subseteq M_{\alpha,\nu}$. Then $y \in [M_{\alpha,\nu} \cap \lambda]^{\mathrm{cf}(\lambda)}$, and so there exists $x \in X_{\alpha,\nu}$ with $x \supseteq y$ by the $\subseteq$-cofinality of $X_{\alpha,\nu}$. This x is as desired. $\square$

Next we examine what happens if $\mathrm{pp}(\lambda) = \lambda^+$ holds for the least singular cardinal λ at which SCH^+ fails:

Lemma 4.3. *Suppose that λ is a singular cardinal such that* $\mathrm{cf}([\nu]^{\mathrm{cf}(\lambda)}, \subseteq) < \lambda$ *for all* $\nu < \lambda$ *and such that* $\mathrm{cf}([\lambda]^{\mathrm{cf}(\lambda)}, \subseteq) > \lambda^+$. *Assume that* $\mathrm{pp}(\lambda) = \lambda^+$. *Let R be the set of all regular cardinals below λ. Then there exists a set $D \subseteq \lambda$ with the following properties:*

(i) $\sup(D) = \lambda$, *and* $|D| = \mathrm{cf}(\lambda)$.

(ii) D consists of limit cardinals of cofinality $> \mathrm{cf}(\lambda)^+$.

(iii) For any $k \in \Pi D$ and any $\mathcal{F} \subseteq \Pi R$ of cardinality $\leq \lambda^+$ there exist a set $A \subseteq R \cap \bigcup_{\nu \in D}[k(\nu), \nu)$ and $g \in \Pi A$ such that

- $\sup(A) = \lambda$, *and* $|A| = \mathrm{cf}(\lambda)$,
- $g \not<^* f \restriction A$ *for any* $f \in \mathcal{F}$.

Proof. Take a sufficiently large regular cardinal θ and a well-ordering Δ of $\mathcal{H}_\theta$, and let $\mathcal{M} := \langle \mathcal{H}_\theta, \in, \Delta, \lambda \rangle$. Take $M \prec \mathcal{M}$ such that $|M| = \mathrm{cf}(\lambda)^+$ and such that M is internally approachable of length $\mathrm{cf}(\lambda)^+$. Let $\langle M_\xi \mid \xi < \mathrm{cf}(\lambda)^+ \rangle$ be an internally approaching sequence to M such that $|M_\xi| \leq \mathrm{cf}(\lambda)$ for each ξ. Moreover let

$$\bar{\mathcal{F}} := \bigcup \{\mathcal{F} \in M \mid \mathcal{F} \subseteq \Pi R \wedge |\mathcal{F}| \leq \lambda^+\} .$$

Note that $\bar{\mathcal{F}} \subseteq \Pi R$ and that $|\bar{\mathcal{F}}| = \lambda^+$. Then by Lem.4.1 we can take an unbounded $A \subseteq R$ and $g \in \Pi A$ such that $|A| = \mathrm{cf}(\lambda)$ and such that $g \not<^* f \restriction A$ for any $f \in \bar{\mathcal{F}}$. Let $A_0 := M \cap A$ and $A_1 := A \setminus M$.

Claim. $\sup(A_1) = \lambda$, *and* $g \restriction A_1 \not<^* f \restriction A_1$ *for any* $f \in \bar{\mathcal{F}}$.

Proof of Claim. Claim is clear if $\sup(A_0) < \lambda$. So assume that $\sup(A_0) = \lambda$.

First we prove that there exists $f_0 \in \bar{\mathcal{F}}$ with $g \restriction A_0 <^* f_0 \restriction A_0$: Take $\xi < \mathrm{cf}(\lambda)^+$ such that $A_0 \subseteq M_\xi$, and let $B := R \cap M_\xi$. Then $\mathrm{tcf}(\Pi B, <^*) = \lambda^+$ by Cor.2.19 and our assumption that $\mathrm{pp}(\lambda) = \lambda^+$. Moreover $B \in M$. Hence we can take $\mathcal{F} \subseteq \Pi R$ in M such that $|\mathcal{F}| = \lambda^+$ and such that for any $h \in \Pi B$ there exists $f \in \mathcal{F}$ with $h <^* f \restriction B$. So there exists $f_0 \in \bar{\mathcal{F}}$ with $g \restriction A_0 <^* f_0 \restriction A_0$.

Then $\sup(A_1) = \lambda$ by the choice of g. Moreover if there exists $f_1 \in \bar{\mathcal{F}}$ such that $g \restriction A_1 <^* f_1 \restriction A_1$, then $\sup\{f_0, f_1\} \in \bar{\mathcal{F}}$, and $g <^* \sup\{f_0, f_1\} \restriction A$. This contradicts the choice of g. Therefore $g \restriction A_1 \not<^* f \restriction A_1$ for any $f \in \bar{\mathcal{F}}$. $\square_{\mathrm{Claim}}$

Let

$$D' := \{\min(M \cap \lambda \setminus \kappa) \mid \kappa \in A_1\} .$$

Note that D' is an unbounded subset of λ of size $\mathrm{cf}(\lambda)$ and that D' consists of limit cardinals of cofinality $> \mathrm{cf}(\lambda)^+$. Take $\eta < \mathrm{cf}(\lambda)^+$ with $D' \subseteq M_\eta$, and let

$$D \ := \ \{\nu \in M_\eta \mid \nu \text{ is a limit cardinal of cofinality} > \mathrm{cf}(\lambda)^+\} \ .$$

Note that $D' \subseteq D \in M$. We show that D witnesses the lemma. It suffices to check the property (iii).

Assume not. Then by the elementarity of M we can take a counter-example $k, \mathcal{F} \in M$ of the property (iii) for D. Note that $k(\nu) \in M \cap \nu$ for each $\nu \in D$ because $k, \nu \in M$. Thus for each $\kappa \in A_1$, if we let $\nu = \min(M \cap \mathrm{On} \setminus \kappa)$, then $\kappa \in [k(\nu), \nu)$. Hence $A_1 \subseteq R \cap \bigcup_{\nu \in D}[k(\nu), \nu)$. Moreover $g \restriction A_1 \not<^* f \restriction A_1$ for any $f \in \mathcal{F}$ by Claim above and the construction of $\bar{\mathcal{F}}$. This contradicts that $k, \mathcal{F}$ is a counter-example. □

Using the previous lemma, next we prove the following. The difference from the previous one is the property (ii) of D:

Lemma 4.4. *Suppose that λ is a singular cardinal such that $\mathrm{cf}([\nu]^{\mathrm{cf}(\lambda)}, \subseteq) < \lambda$ for all $\nu < \lambda$ and such that $\mathrm{cf}([\lambda]^{\mathrm{cf}(\lambda)}, \subseteq) > \lambda^+$. Assume that $\mathrm{pp}(\lambda) = \lambda^+$. Let R be the set of all regular cardinals below λ. Then there exists a set $D \subseteq \lambda$ with the following properties:*

(i) $\sup(D) = \lambda$, *and* $|D| = \mathrm{cf}(\lambda)$.

(ii) D consists of limit cardinals, and $\sup_{\nu \in D} \mathrm{cf}(\nu) < \lambda$.

(iii) For any $k \in \Pi D$ and any $\mathcal{F} \subseteq \Pi R$ of cardinality $\leq \lambda^+$ there exist $A \subseteq R \cap \bigcup_{\nu \in D}[k(\nu), \nu)$ and $g \in \Pi A$ such that

- $\sup(A) = \lambda$, *and* $|A| = \mathrm{cf}(\lambda)$,
- *$g \not<^* f \restriction A$ for any $f \in \mathcal{F}$.*

For this we use the following lemma:

Lemma 4.5. *Let λ be a singular cardinal, A be a set of regular cardinals with $|A| < \lambda = \sup(A)$ and μ be a regular cardinal with $|A| < \mu < \lambda$. Suppose that $\vec{f} = \langle f_\alpha \mid \alpha < \lambda^+ \rangle$ is a scale in $\langle \Pi A, <^* \rangle$. Then there are stationary many $\alpha \in E^{\lambda^+}_\mu$ such that $\langle f_\beta \mid \beta < \alpha \rangle$ has the exact upper bound g with respect to $<^*$ with $\mathrm{cf}(g(\kappa)) = \mu$ for all $\kappa \in A$.*

Proof. If $\alpha \in E^{\lambda^+}_\mu$ is a good point for $\vec{f}$, and $b \subseteq \alpha$ and $\nu < \lambda$ witness goodness of α, then, using the fact that $|A| < \mu$, it is easy to see that the function $g \in {}^A\mathrm{On}$ defined as

$$g(\kappa) \ = \ \begin{cases} \sup_{\beta \in b} f_\beta(\kappa) & \cdots \quad \text{if } \kappa \in A \setminus \nu \\ \mu & \cdots \quad \text{if } \kappa \in A \cap \nu \end{cases}$$

is the exact upper bound of $\langle f_\beta \mid \beta < \alpha \rangle$. Moreover $\mathrm{cf}(g(\kappa)) = \mu$ for all $\kappa \in A$. Then Lem.4.5 follows from Lem.2.8. □

Proof of Lem.4.4. For the contradiction assume not.

Let D be a subset of λ obtained by Lem.4.3, and let $B := \{\mathrm{cf}(\nu) \mid \nu \in D\}$. Note that $|B| < \lambda = \sup(B)$. Then $\mathrm{tcf}(\Pi B, <^*) = \lambda^+$ by Cor.2.19 and the assumption that $\mathrm{pp}(\lambda) = \lambda^+$. Let $\vec{h}' = \langle h'_\alpha \mid \alpha < \lambda^+ \rangle$ be a scale in $\langle \Pi B, <^* \rangle$.

By Lem.4.5 we may assume that there are stationary many $\alpha \in E^{\lambda^+}_{\mathrm{cf}(\lambda)^+}$ such that h'_α is the exact upper bound of $\langle h'_\beta \mid \beta < \alpha \rangle$ with respect to $<^*$ and such that $\mathrm{cf}(h'_\alpha(\kappa)) = \mathrm{cf}(\lambda)^+$ for all $\kappa \in B$. Let E be the set of all such α.

Next for each $\nu \in D$ take an increasing continuous sequence $\langle \delta_{\nu,\gamma} \mid \gamma < \mathrm{cf}(\nu) \rangle$ of cardinals $< \nu$ which converges to ν and such that $\langle \delta_{\nu,0} \mid \nu \in D \rangle$ converges to λ. We can take such sequences because D is nonstationary in λ. Then for each $\alpha < \lambda^+$ define $h_\alpha \in \Pi D$ by $h_\alpha(\nu) = \delta_{\nu, h'_\alpha(\mathrm{cf}(\nu))}$. Moreover let I be the ideal over D consisting of all $D' \subseteq D$ such that $\{\mathrm{cf}(\nu) \mid \nu \in D'\}$ is bounded in λ. Note the following:

- $\vec{h} := \langle h_\alpha \mid \alpha < \lambda^+ \rangle$ is a scale in $\langle \Pi D, <_I \rangle$.
- For each $\alpha \in E$, h_α is the exact upper bound of $\langle h_\beta \mid \beta < \alpha \rangle$ with respect to $<_I$.
- For each $\alpha \in E$, $h_\alpha(\nu)$ is a limit cardinal of cofinality $\mathrm{cf}(\lambda)^+$ for all $\nu \in D$.

Because Lem.4.4 fails, for each $\alpha \in E$ the set $\{h_\alpha(\nu) \mid \nu \in D\}$ does not satisfies the property (iii) in Lem.4.4. Hence for each $\alpha \in E$ we can take $k^0_\alpha \in \Pi D$ and $\mathcal{F}^0_\alpha \subseteq \Pi R$ such that

- $k^0_\alpha < h_\alpha$,
- for any $A \subseteq R \cap \bigcup_{\nu \in D} [k^0_\alpha(\nu), h_\alpha(\nu))$ with $\sup(A) = \lambda$ and $|A| = \mathrm{cf}(\lambda)$ and for any $g \in \Pi A$ there exists $f \in \mathcal{F}^0_\alpha$ with $g <^* f \restriction A$.

Then for each $\alpha \in E$ we can also take $\gamma_\alpha < \alpha$ such that $k^0_\alpha <_I h_{\gamma_\alpha}$. By Fodor's lemma take $\gamma < \lambda^+$ such that the set $\{\alpha \in E \mid \gamma_\alpha = \gamma\}$ is stationary. Let $\mathcal{F}^0 := \bigcup \{\mathcal{F}^0_\alpha \mid \alpha \in E \wedge \gamma_\alpha = \gamma\}$.

Moreover take a cofinal $C \subseteq \lambda$ of order-type $\mathrm{cf}(\lambda)$, and for each $\mu \in C$ let $D_\mu := \{\nu \in D \mid \mathrm{cf}(\nu) < \mu\}$. Because Lem.4.4 fails, for each $\mu \in C$ we can take $k^1_\mu \in \Pi D_\mu$ and $\mathcal{F}^1_\mu \subseteq \Pi R$ of cardinality $\leq \lambda^+$ such that

- for any $A \subseteq R \cap \bigcup_{\nu \in D_\mu} [k^1_\mu(\nu), \nu)$ with $\sup(A) = \lambda$ and $|A| = \mathrm{cf}(\lambda)$ and for any $g \in \Pi A$ there exists $f \in \mathcal{F}^1_\mu$ with $g <^* f \restriction A$.

Let $\mathcal{F}^1 := \bigcup\{\mathcal{F}^1_\mu \mid \mu \in C\}$.

Let $\mathcal{F} := \{\sup\{f^0, f^1\} \mid f^0 \in \mathcal{F}^0 \wedge f^1 \in \mathcal{F}^1\}$. Moreover take $k \in \Pi D$ such that $h_\gamma < k$ and such that $k^1_\mu < k \restriction D_\mu$ for all $\mu \in C$. We can take such k because $\mathrm{cf}(\nu) > \mathrm{cf}(\lambda)^+$ for all $\nu \in D$.

Because D witnesses Lem.4.3, we can take a set $A \subseteq R \cap \bigcup_{\nu\in D}[k(\nu), \nu)$ and $g \in \Pi A$ such that

- $\sup(A) = \lambda$, and $|A| = \mathrm{cf}(\lambda)$,
- $g \not<^* f \restriction A$ for any $f \in \mathcal{F}$.

Here note that $\sup(A \cap \nu) < \nu$ for all $\nu \in D$ because $|A| = \mathrm{cf}(\lambda) < \mathrm{cf}(\nu)$. Then we can take $\alpha \in E$ such that

- $\gamma_\alpha = \gamma$,
- $\{\nu \in D \mid h_\alpha(\nu) \leq \sup(A \cap \nu)\} \in I$.

Take $\mu < \lambda$ so that $k^0_\alpha(\nu) < h_\gamma(\nu) < h_\alpha(\nu)$ and $\sup(A \cap \nu) < h_\alpha(\nu)$ for all $\nu \in D \setminus D_\mu$. Let

$$A^0 \ := \ A \cap \bigcup_{\nu\in D\setminus D_\mu}[k(\nu), \nu) \ \subseteq \ \bigcup_{\nu\in D\setminus D_\mu}[k^0_\alpha(\nu), h_\alpha(\nu)) \ .$$

Then we can take $f^0 \in \mathcal{F}^0_\alpha$ and $\rho^0 < \lambda$ such that $g \restriction A^0 <_{\rho^0} f^0 \restriction A^0$ by the choice of k^0_α and $\mathcal{F}^0_\alpha$.

Next let

$$A^1 \ := \ A \cap \bigcup_{\nu\in D_\mu}[k(\nu), \nu) \ .$$

Then we can take $f^1 \in \mathcal{F}^1_\mu$ and $\rho^1 < \lambda$ such that $g \restriction A^1 <_{\rho^1} f^1 \restriction A^1$.

Let $f := \sup\{f^0, f^1\}$ and $\rho := \max\{\rho^0, \rho^1\}$. Note that $f \in \mathcal{F}$ and that $\rho < \lambda$. Moreover $g <_\rho f \restriction A$. This contradicts that the choice of A and g. □

Now we prove Thm.1.2:

Proof of Thm.1.2. For the contradiction assume that λ is the least singular cardinal at which SCH^+ fails and that $\mathrm{pp}(\lambda) = \lambda^+$. Let R be the set of all regular cardinals below λ.

Then λ satisfies the assumption of Lem.4.4. Let D be a set obtained by Lem.4.4. Then it is easy to see that

$$\mu \ := \ \mathrm{cf}(\Pi D, <) \ \leq \ \mathrm{cf}([\sup_{\nu\in D}\mathrm{cf}(\nu)]^{\mathrm{cf}(\lambda)}, \subseteq) \ < \ \lambda \ .$$

By reducing D if necessary, we may assume that $\min(D) > \mu$. Take a $<$-cofinal $\mathcal{K} \subseteq \Pi D$ of cardinality μ such that $k(\nu) > \mu$ for all $k \in \mathcal{K}$ and all $\nu \in D$.

Note that $\mathrm{cf}([\nu]^\mu, \subseteq) \leq \nu^+$ for all $\nu \in D$ by Lem.3.2. Then for each $\nu \in D$ we can take a $<^*$-increasing sequence $\vec{h}_\nu = \langle d_{\nu,\gamma} \mid \gamma < \nu^+ \rangle$ in $\Pi(R \cap \nu)$ such that for any $A \subseteq R \cap \nu$ of cardinality $\leq \mu$ and any $g \in \Pi A$ there exists $\gamma < \nu^+$ with $g <^* d_{\nu,\gamma} \restriction A$.

Next note that $\mathrm{tcf}(\Pi D^+, <^*) = \lambda^+$ by Cor.2.19 and the assumption that $\mathrm{pp}(\lambda) = \lambda^+$. Here D^+ denotes the set $\{\nu^+ \mid \nu \in D\}$. Let $\langle e_\alpha \mid \alpha < \lambda^+ \rangle$ be a scale in $\langle \Pi D^+, <^* \rangle$. Moreover for each $\alpha < \lambda^+$ define $f_\alpha \in \Pi R$ as

$$f_\alpha(\kappa) \;=\; \begin{cases} \sup\{d_{\nu, e_\alpha(\nu^+)}(\kappa) \mid \kappa < \nu \in D\} & \cdots \quad \text{if } \kappa > \mathrm{cf}(\lambda) \\ 0 & \cdots \quad \text{otherwise} \end{cases}$$

for each $\kappa \in R$.

By the choice of D, for each $k \in \mathcal{K}$ we can take $A_k \subseteq R \cap \bigcup_{\nu \in D} [k(\nu), \nu)$ and $g_k \in \Pi A_k$ such that $\sup(A_k) = \lambda$, such that $|A_k| = \mathrm{cf}(\lambda)$ and such that $g_k \not<^* f_\alpha \restriction A_k$ for any $\alpha < \lambda^+$. Let $A := \bigcup_{k \in \mathcal{K}} A_k$, and define $g \in \Pi A$ as

$$g(\kappa) \;=\; \sup\{g_k(\kappa) \mid k \in \mathcal{K} \wedge \kappa \in A_k\}$$

for each $\kappa \in A$.

Take $e \in \Pi D^+$ such that $g \restriction A \cap \nu <^* d_{\nu, e(\nu)} \restriction A \cap \nu$ for each $\nu \in D$. Moreover take $\alpha < \lambda^+$ such that $e <^* e_\alpha$. Let $\rho < \lambda$ be such that $e <_\rho e_\alpha$. Then $g \restriction A \cap \nu <^* f_\alpha \restriction A \cap \nu$ for each $\nu \in D \setminus \rho$. Take $k \in \mathcal{K}$ such that $g \restriction A \cap \nu <_{k(\nu)} f_\alpha \restriction A \cap \nu$ for each $\nu \in D \setminus \rho$.

Here recall that $g_k \leq g$ and that $A_k \subseteq R \cap \bigcup_{\nu \in D} [k(\nu), \nu)$. Hence $g_k <_\rho f_\alpha \restriction A_k$. This contradicts the choice of A_k and g_k. □

5 Better scale

In this section we prove Thm.1.3.

Theorem 1.3. *Assume that* SCH^+ *fails, and let* λ *be the least singular cardinal at which* SCH^+ *fails. Then there exists a set* A *of regular cardinals below* λ *such that*

(i) $\mathrm{o.t.}(A) = \omega$, *and* $\sup(A) = \lambda$,

(ii) $\langle \Pi A, <^* \rangle$ *has a better scale of length* λ^+.

By Thm.1.2 it suffices to prove the following:

Proposition 5.1. *Suppose that* λ *is a singular cardinal with* $\mathrm{pp}(\lambda) > \lambda^+$. *Then there exists a set* A *of regular cardinals such that* *(i)* o.t.$(A) = \mathrm{cf}(\lambda)$*, and* $\sup(A) = \lambda$*,*

(ii) $\langle \Pi A, <^* \rangle$ *has a better scale of length* λ^+*.*

Proof. By Lem.2.16 and the fact that $\mathrm{pp}(\lambda) > \lambda^+$ we can take a set B of regular cardinals such that $\sup(B) = \lambda$, such that o.t.$(B) = \mathrm{cf}(\lambda)$ and such that $\langle \Pi B, <^* \rangle$ is $\leq \lambda^+$-directed. Moreover take a club $c_\alpha \subseteq \alpha$ for each $\alpha \in \mathrm{Lim}(\lambda^+)$.

Because $\langle \Pi B, <^* \rangle$ is $\leq \lambda^+$-directed, we can inductively construct a $<^*$-increasing sequence $\vec{f} = \langle f_\beta \mid \beta < \lambda^+ \rangle$ in ΠB so that the following holds for all $\beta < \lambda^+$:

$(*)$ $\sup\{f_\gamma \mid \gamma \in c_\alpha \cap \beta\} <^* f_\beta$ for all $\alpha < \lambda^+$.

For each $\alpha \in \mathrm{Lim}(\lambda^+)$ let $\sigma_\alpha : c_\alpha \to \lambda$ be the function such that

$$\sup\{f_\gamma \mid \gamma \in c_\alpha \cap \beta\} <_{\sigma_\alpha(\beta)} f_\beta \, .$$

Then c_α and σ_α witnesses that α is a better point for $\vec{f}$. In particular, every $\alpha \in E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ is a better (hence good) point for $\vec{f}$.

Then by Lem.2.8 $\vec{f}$ has the exact upper bound f with respect to $<^*$ such that $\langle \mathrm{cf}(f(\kappa)) \mid \kappa \in B \rangle$ converges to λ. Take an unbounded $B' \subseteq B$ with $\langle \mathrm{cf}(f(\kappa)) \mid \kappa \in B' \rangle$ strictly increasing. Moreover let $A := \{\mathrm{cf}(f(\kappa)) \mid \kappa \in B'\}$.

For each $\kappa \in B'$ take a club $D_\kappa \subseteq f(\kappa)$ of order-type $\mathrm{cf}(f(\kappa))$. Moreover for each $\beta < \lambda^+$ let $g_\beta \in \Pi A$ be such that for each $\kappa \in B'$, $g_\beta(\mathrm{cf}(\kappa)) =$ o.t.$(f_\beta(\kappa) \cap D_\kappa)$ if $f_\beta(\kappa) < f(\kappa)$.

Note that $\langle g_\beta \mid \beta < \lambda^+ \rangle$ is $\leq^*$-increasing $<^*$-cofinal sequence in ΠA. Hence we can take a club $E \subseteq \lambda^+$ such that $\langle g_\beta \mid \beta \in E \rangle$ is $<^*$-increasing and $<^*$-cofinal in ΠA. Let $\langle \beta_\alpha \mid \alpha < \lambda^+ \rangle$ be the increasing enumeration of E, and let $h_\alpha := g_{\beta_\alpha}$. Then it is easy to see that $\langle h_\alpha \mid \alpha < \lambda^+ \rangle$ is a better scale in $\langle \Pi A, <^* \rangle$. □

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