Π^1_{n+1}-indescribability and simultaneous reflection of Π^1_n-positive sets

Hiroshi Sakai

1 Intoroduction

In this note we study the relationship between the Π^1_{n+1}-indescribability of cardinals and the reflection of Π^1_n-positive sets. First recall the Π^1_n-indescribability and the Π^1_n-positivity:

Definition 1.1. *Let $n \in \omega$, and let κ be an ordinal. $S \subseteq \kappa$ is said to be Π^1_n-*positive in κ *if for any $A \subseteq V_\kappa$ and any Π^1_n-formula φ with $V_\kappa \models \varphi[A]$ there exists $\mu \in S$ such that $V_\mu \models \varphi[A \cap V_\mu]$. κ is said to be Π^1_n-indescribable if κ itself is Π^1_n-positive in κ.*

Fact 1.2 ((1):Levy [3], (2):Hanf-Scott [2], (3):Levy [4])**.**

(1) An ordinal κ is Π^1_0-indescribable if and only if κ is an inaccessible cardinal.

(2) An ordinal κ is Π^1_1-indescribable if and only if κ is a weakly compact cardinal.

(3) Suppose that κ is Π^1_0-indescribable. Then $S \subseteq \kappa$ is Π^1_0-positive if and only if S is stationary.

In this note we give a characterization Π^1_{n+1}-indescribability using simultaneous reflection of Π^1_n-positive set:

Theorem 1.3. *Let $n \in \omega$. Then the following are equivalent for any Π^1_n-indescribable cardinal κ:*

(I) κ is Π^1_{n+1}-indescribable.

(II) For any function $F : {}^{<\kappa}2 \to 2$, if the set $\{\mu < \kappa \mid F(B \restriction \mu) = 1\}$ is Π^1_n-positive in κ for every $B \in {}^{\kappa}2$, then there exists $\lambda < \kappa$ such that the set $\{\mu < \lambda \mid F(B \restriction \mu) = 1\}$ is Π^1_n-positive in λ for every $B \in {}^{\lambda}2$.

We also prove that for $n = 0$ the Π^1_1-indescribability is not equivalent to the diagonal simultaneous reflection of Π^1_0-positive (i.e. stationary) sets:

Theorem 1.4. *Suppose that κ is a Π^1_1-indescribable cardinal. Then there exists a forcing extension in which the following hold:*

(I) κ is Π^1_0-indescribable but not Π^1_1-indescribable.

(II) For any sequence $\langle S_\alpha \mid \alpha < \kappa \rangle$ of Π^1_0-positive subsets of κ there exists $\lambda < \kappa$ such that $S_\alpha \cap \lambda$ is Π^1_0-positive in λ for all $\alpha < \lambda$.

We conjecture that the above theorem can be generalized to the following:

Conjecture 1.5. *Suppose that $n \in \omega$ and that κ is a Π^1_{n+1}-indescribable cardinal. Then there exists a forcing extension in which the following hold:*

(I) κ is Π^1_n-indescribable but not Π^1_{n+1}-indescribable.

(II) For any sequence $\langle S_\alpha \mid \alpha < \kappa \rangle$ of Π^1_n-positive subsets of κ there exists $\lambda < \kappa$ such that $S_\alpha \cap \lambda$ is Π^1_n-positive in λ for all $\alpha < \lambda$.

2 Preliminaries

Here we present our notation and basic facts used in this note. For those which are not presented below, consult Kanamori [5] or Drake [1].

First we give our notation relevant to the first order logic. Let ZFC^- be $\mathsf{ZFC} - \mathsf{Power\ Set\ Axiom}$. As is the custom, a structure $\langle M, \in \rangle$ for some set M is simply denoted as M. Let $\mathcal{M}$ and $\mathcal{N}$ be structures. Then $\mathcal{M} \prec \mathcal{N}$ denotes that $\mathcal{M}$ is an elementary submodel of $\mathcal{N}$ with respect to first order formulae (and not second order formulae).

Next we give our notation and basic facts relevant to the second order logic. In second order formulae we use lower-case characters $x, y, z, \ldots$ for first order variables and upper-case characters $X, Y, Z, \ldots$ for second order variables. As in the case with the first order logic, a structure $\langle M, \in \rangle$ for some set M is simply denoted as M.

In this note we frequently use the following fact without any notices: Let $\varphi(X_0, \ldots, X_{l-1})$ be a Π^1_n-formula for some $n \in \omega$. Then there is a Π^1_n-formula $\varphi'(X)$ such that for any uncountable cardinal κ and any $A_0, \ldots, A_{l-1} \subseteq V_\kappa$, letting $A := \bigcup_{i<l} \{i\} \times A_i$, it holds that $V_\kappa \models \varphi[A_0, \ldots, A_{l-1}]$ if and only if $V_\kappa \models \varphi'[A]$.

We also use the following fact on the existence of a universal formula, whose proof can be found in [1]:

Fact 2.1. *Suppose that $1 \leq n < \omega$, and let Γ be Π^1_n or Σ^1_n. Then there is a Γ-formula $\Phi(x, X)$ with the following property: For any Γ-formula $\varphi(X)$ there is $l < \omega$ such that*

$$V_\kappa \models \varphi[A] \;\Leftrightarrow\; V_\kappa \models \Phi[l, A]$$

for all uncountable cardinal κ and all $A \subseteq V_\kappa$.

$\Phi(x, X)$ as in Fact 2.1 is called a *universal* Γ*-formula*.

3 Characterization of Π^1_{n+1}-indescribability using simultaneous reflection of Π^1_n-positive sets

In this section we prove Thm.1.3. In fact we prove the following which implies Thm.1.3:

Theorem 3.1. *Let $n \in \omega$, and suppose that κ is a Π^1_n-indescribable cardinal. Then the following are equivalent for any $T \subseteq \kappa$:*

(I) T is Π^1_{n+1}-positive.

(II) For any function $F : {}^{<\kappa}2 \to 2$, if the set $\{\mu < \kappa \mid F(B \restriction \mu) = 1\}$ is Π^1_n-positive in κ for every $B \in {}^{\kappa}2$, then there exists $\lambda \in T$ such that the set $\{\mu < \lambda \mid F(B \restriction \mu) = 1\}$ is Π^1_n-positive in λ for any $B \in {}^{\lambda}2$.

Proof. Let T be a subset of κ.

First we prove that (I) implies (II). Assume (I). Suppose that F is a function from ${}^{<\kappa}2$ to 2 and that the set $\{\mu < \kappa \mid F(B \restriction \mu) = 1\}$ is Π^1_n-positive for every $B \in {}^{\kappa}2$. We will find $\lambda < \kappa$ witnessing (II) for F.

First suppose that $n = 0$. Recalling Fact 1.2, let $\varphi(X)$ be a Π^1_1-formula representing the following:

- A class of ordinals is an inaccessible cardinal.
- $\forall Y \in {}^{\mathrm{On}}2$, $\{\mu \in \mathrm{On} \mid X(Y \restriction \mu) = 1\}$ is stationary.

Then $V_\kappa \models \varphi[F]$. By the Π^1_1-positivity of T there is $\lambda \in T$ with $V_\lambda \models \varphi[F \restriction {}^{<\lambda}2]$. Then λ is as desired.

Next suppose that $n > 0$. Let $\Phi(z, Z)$ be a universal Π^1_n-formula and $\varphi(X)$ be a Π^1_{n+1}-formula representing the following:

- $\forall Y \in {}^{\mathrm{On}}2\, \forall Z\, \forall z \in \omega$,
  $[\Phi(z, Z) \to \{\exists \mu \in \mathrm{On},\ X(Y \restriction \mu) = 1 \wedge V_\mu \models \Phi(z, Z \cap V_\mu)\}]$.

Then $V_\kappa \models \varphi[F]$. By the Π^1_{n+1}-indescribability of T there is $\lambda \in T$ such that $V_\lambda \models \varphi[F \restriction {}^{<\lambda}2]$. Then λ is as desired.

Next we prove that (II) implies (I). Assume (II). Suppose that $\varphi(X)$ is a Π^1_{n+1}-formula, that $A \subseteq V_\kappa$ and that $V_\kappa \models \varphi[A]$. We must find $\lambda \in T$ with $V_\lambda \models \varphi[A \cap V_\lambda]$. Let $\psi(X, Y)$ be the Σ^1_n-formula such that $\varphi(X) \equiv \forall Y,\ \psi(X, Y)$.

First note that κ is an inaccessible cardinal because it is Π^1_n-indescribable. Take a bijection $f : V_\kappa \to \kappa$, and let C be the set of all $\mu < \kappa$ such that $f \restriction V_\mu$ is a bijection from V_μ to μ. Note that C is club in κ. For each $\mu \in C \cup \{\kappa\}$ and each $B \in {}^{\mu}2$ let $\hat{B} := \{a \in V_\mu \mid B(f(a)) = 1\}$.

Then define $F : {}^{<\kappa}2 \to 2$ as follows:

- If $\mu \in \kappa \setminus C$, then $F(B) = 0$ for any $B \in {}^{\mu}2$.
- Suppose that $\mu \in C$ and that $B \in {}^{\mu}2$. $F(B) = 1$ if $V_\mu \models \psi[A \cap V_\mu, \hat{B}]$, and $F(B) = 0$ otherwise.

Note that the following holds for each $B \in {}^{\kappa}2$:

$$\{\mu < \kappa \mid F(B \restriction \mu) = 1\} \ = \ C \cap \{\mu < \kappa \mid V_\mu \models \psi[A \cap V_\mu, \hat{B} \cap V_\mu]\}\ .$$

Then the set $\{\mu < \kappa \mid F(B \restriction \mu) = 1\}$ is Π^1_n-positive for all $B \in {}^{\kappa}2$ because κ is Π^1_n-indescribable, and $V_\kappa \models \varphi[A]$. So by (II) there exists $\lambda \in T$ such that the set $\{\mu < \lambda \mid F(B \restriction \mu) = 1\}$ is Π^1_n-positive for all $B \in {}^{\lambda}2$. We claim that $V_\lambda \models \varphi[A \cap V_\lambda]$.

Assume not. Then there is $D \subseteq V_\lambda$ such that $V_\lambda \models \neg\psi[A \cap V_\lambda, D]$. Let $B \in {}^{\lambda}2$ be such that $D = \hat{B}$. Then, because there are Π^1_n-positively many $\mu < \lambda$ with $F(B \restriction \mu) = 1$, we can take $\mu < \lambda$ such that $F(B \restriction \mu) = 1$ and such that $V_\mu \models \neg\psi[A \cap V_\mu, D \cap V_\mu]$. But the fact that $F(B \restriction \mu) = 1$ implies that $V_\mu \models \psi[A \cap V_\mu, D \cap V_\mu]$. This is a contradiction. □

4 Separation of Π^1_1-indescribability and diagonal simultaneous reflection of stationary sets

In this section we prove Thm.1.4, which can be also stated as follows:

Theorem 4.1. *Assume that a κ is Π^1_1-indescribable cardinal. Then there is a forcing extension in which the following holds:*

(I) *κ is inaccessible but not Π^1_1-indescribable.*

(II) *For any sequence $\langle S_\xi \mid \xi < \kappa\rangle$ of stationary subsets of κ there exists an inaccessible $\lambda < \kappa$ such that $S_\xi \cap \lambda$ is stationary in λ for all $\xi < \lambda$.*

Roughly speaking, we construct a forcing extension in which the following holds:

(III) κ remains to be inaccessible, and there is a function $F : {}^{<\kappa}2 \to 2$ such that for any $B \in {}^{\kappa}2$ the set $\{\mu < \kappa \mid F(B \restriction \mu) = 1\}$ is stationary and such that for any $\lambda < \kappa$ there is $B \in {}^{\lambda}2$ with the set $\{\mu < \lambda \mid F(B \restriction \mu) = 1\}$ is non-stationary.

(IV) There is a poset, $\mathbb{C}^*(F)$, which preserves stationary subsets of κ and forces that κ is Π^1_1-indescribable.

The properties (III) and (IV) above imply (I) and (II) in Thm.4.1, respectively.

First we give a poset $\mathbb{C}^*(F)$ in (IV):

Definition 4.2. *Suppose that κ is an inaccessible cardinal and that F is a function from ${}^{<\kappa}2$ to 2.*

Let $\mathbb{C}(F)$ be the poset of all pairs (b, c) such that

(i) $b \in {}^{<\kappa}2$,

(ii) *c is a closed bounded subset of κ such that* $\max(c) = \mathrm{dom}(b)$,

(iii) $F(b \restriction \mu) = 0$ *for any $\mu \in c$.*

$(b_0, c_0) \leq (b_1, c_1)$ in $\mathbb{C}(F)$ if $b_0 \supseteq b_1$, and c_0 is an end-extension of c_1. For $p = (b, c) \in \mathbb{C}(F)$ we let ρ_p denote $\mathrm{dom}(b)$ *(*$= \max(c)$*).*

Let $\mathbb{C}^*(F)$ be the $< \kappa$-support product of κ^+-many copies of $\mathbb{C}(F)$. That is, $\mathbb{C}^*(F)$ is the poset of all partial functions $p : \kappa^+ \to \mathbb{C}(F)$ with $|\mathrm{dom}(p)| < \kappa$. $p_0 \leq p_1$ in $\mathbb{C}^*(F)$ if $\mathrm{dom}(p_0) \supseteq \mathrm{dom}(p_1)$, and $p_0(\alpha) \leq p_1(\alpha)$ for all $\alpha \in \mathrm{dom}(p_1)$.

Lemma 4.3. *Suppose that κ is an inaccessible cardinal and that $F : {}^{<\kappa}2 \to 2$. Then $\mathbb{C}^*(F)$ has the κ^+-c.c.*

Proof. By the standard argument using the Δ-system lemma. □

Lemma 4.4. *Suppose that κ is an inaccessible cardinal and that $F : {}^{<\kappa}2 \to 2$.*

(1) *Suppose that $F(b) = 0$ for all $b \in {}^{<\kappa}2$ such that* $\mathrm{dom}(b)$ *is not inaccessible. Then $\mathbb{C}^*(F)$ has a $<\lambda$-closed dense subset for any $\lambda < \kappa$.*

(2) *Suppose that there is a club $C \subseteq \kappa$ such that $F(b) = 0$ for all $b \in {}^{<\kappa}2$ such that $\mathrm{dom}(b) \in C$. Then $\mathbb{C}^*(F)$ has a $<\kappa$-closed dense subset.*

Proof. (1) Fix $\lambda < \kappa$. Then $D_\lambda := \{p \in \mathbb{C}^*(F) \mid \forall\alpha \in \mathrm{dom}(p),\ \rho_{p(\alpha)} > \lambda\}$ is a $<\lambda$-closed dense subset of $\mathbb{C}^*(F)$.

(2) It is easy to see that $D := \{p \in \mathbb{C}^*(F) \mid \forall\alpha \in \mathrm{dom}(p),\ \rho_{p(\alpha)} \in C\}$ is a $<\kappa$-closed dense subset of $\mathbb{C}^*(F)$. □

Lemma 4.5. *Let κ be an inaccessible cardinal and $F : {}^{<\kappa}2 \to 2$ be a function with the following property:*

- *$F(b) = 0$ for all $b \in {}^{<\kappa}2$ such that $\mathrm{dom}(b)$ is not inaccessible.*
- *For each inaccessible $\mu < \kappa$ there is at most one $b \in {}^{\mu}2$ with $F(b) = 1$.*
- *Suppose that $\mu < \kappa$ is inaccessible, that N is a transitive model of ZFC^- with $|N| = \mu$ and V_μ, $F \restriction {}^{<\mu}2 \in N$ and that $q \in \mathbb{C}^*(F \restriction {}^{<\mu}2)^N$. Then there exists a $\mathbb{C}^*(F \restriction {}^{<\mu}2)^N$-generic filter over N containing q.*

Then $\mathbb{C}^*(F)$ preserves stationary subsets of κ.

Proof. First note that the Approachability Property at κ holds because κ is (strongly) inaccessible. Hence, by Lem.4.4 (1), $\mathbb{C}^*(F)$ preserves stationary sets consisting of singular cardinals. Thus it suffices to show that $\mathbb{C}^*(F)$ preserves stationary consisting of inaccessible cardinals.

Take an arbitrary stationary $S \subseteq \kappa$ consisting of inaccessible cardinals. Suppose that $q \in \mathbb{C}^*(F)$ and that $\dot{C}$ is a $\mathbb{C}^*(F)$-name of a club subset of κ. We find $p \leq q$ and $\mu \in S$ such that $p \Vdash$ " $\mu \in \dot{C}$ ".

Let θ be a sufficiently large regular cardinal, and take $M \prec \langle \mathcal{H}_\theta, \in \rangle$ such that $\kappa, F, p, \dot{C} \in M$ and such that $\mu := M \cap \kappa = |M| \in S$. Note that $V_\mu \subseteq M$. Let $\sigma : M \to N$ be the transitive collapse, and let $\mathbb{C}' := \sigma(\mathbb{C}^*(F)) = \mathbb{C}^*(F \restriction {}^{<\mu}2)^N$.

Claim. *There exists a $\mathbb{C}'$-generic filter I such that for all $\beta \in (\mu^+)^N$ if we let $b_\beta := \bigcup\{b \mid \exists p \in I \exists c,\ (b, c) \in p(\beta)\}$, then $F(b_\beta) = 0$.*

Proof of Claim. First note that $\mathbb{C}' \times \mathbb{C}' \cong \mathbb{C}'$. Hence by the assumption in the lemma we can take a $\mathbb{C}' \times \mathbb{C}'$-filter $I_0 \times I_1$ over N with $q \in I_0, I_1$. Then, because there exists at most one $b \in {}^{\mu}2$ with $F(b) = 1$, either I_0 or I_1 witnesses the claim. $\square_{\text{Claim}}$

Let I and $\langle b_\beta \mid \beta \in (\mu^+)^N \rangle$ be as in Claim. Moreover for each $\beta \in (\mu^+)^N$ let $c_\beta := (\bigcup\{c \mid \exists p \in I \exists b,\ (b, c) \in p(\beta)\}) \cup \{\mu\}$.

Let p be the function on $\kappa^+ \cap M$ such that $p(\alpha) = (b_{\sigma(\alpha)}, c_{\sigma(\alpha)})$. Then it is easy to see that $p \in \mathbb{C}^*(F)$. Moreover $p \Vdash$ " $\mu \in \dot{C}$ " by the genericity of I. Thus p and μ are as desired. $\square$

Next we give a poset adding F in the property (III):

Definition 4.6. *For an inaccessible cardinal κ let $\mathbb{R}(\kappa)$ be the poset of all $p : {}^{\leq \upsilon_p}2 \to 2$ for some $\upsilon_p < \kappa$ with the following properties:*

(i) *$p(b) = 0$ for any $b \in {}^{\mu}2$ such that $\mathrm{dom}(b)$ is not inaccessible.*

(ii) *For any inaccessible $\mu \leq \upsilon_p$ there is at most one $b \in {}^{\mu}2$ with $p(b) = 1$.*

(iii) *Suppose that $\mu \leq \upsilon_p$ is inaccessible, that N is a transitive model of ZFC^- with $|N| = \mu$ and $V_\mu, p \restriction \mu \in N$ and that $q \in \mathbb{C}^*(p \restriction \mu)^N$. Then there exists an $\mathbb{C}^*(p \restriction \mu)^N$-generic filter over N containing q.*

$p_0 \leq p_1 \in \mathbb{R}(\kappa)$ if $p_0 \supseteq p_1$.

Lemma 4.7. *Suppose that κ is an inaccessible cardinal. Then $\mathbb{R}(\kappa)$ has a $<\lambda$-closed dense subset for any $\lambda < \kappa$.*

Proof. Fix $\lambda < \kappa$. Let $D := \{p \in \mathbb{R}(\kappa) \mid \upsilon_p > \lambda\}$. Then it is easy to see that D is $<\lambda$-closed.

To see the density of D, suppose that $q \in \mathbb{R}$. We find $p \in D$ with $p \leq q$. We may assume that $\upsilon_q \leq \lambda$. Let p be a function from ${}^{\leq \lambda+1}2$ to 2 such that $p \restriction {}^{\leq \upsilon_q}2 = q$ and such that $p(b) = 0$ for all $b \in {}^{\leq \lambda+1}2 \setminus {}^{\leq \upsilon_q}2$. Then, using Lem.4.4 (2), we can easily prove that $p \in \mathbb{R}(\kappa)$. Then $p \leq q$, and $p \in D$. $\square$

Lemma 4.8. *Suppose that κ is a Mahlo cardinal in V. Then in $V^{\mathbb{R}(\kappa)}$ for any $B \in {}^{\kappa}2$ the set $\{\mu < \kappa \mid \bigcup \dot{H}(B \restriction \mu) = 1\}$ is stationary in κ, where $\dot{H}$ is the canonical name for a $\mathbb{R}(\kappa)$-generic filter.*

Proof. Let $\dot{B}$ be an $\mathbb{R}(\kappa)$-name for an element of ${}^{\kappa}2$, and let $\dot{C}$ be an $\mathbb{R}(\kappa)$-name for a club subset of κ. Suppose also that $q \in \mathbb{R}(\kappa)$. We find $p \leq q$ and $\mu < \kappa$ such that

$$p \Vdash \text{“ } \bigcup \dot{H}(\dot{B} \restriction \mu) = 1 \ \wedge \ \mu \in \dot{C} \text{ ”} .$$

Let θ be a sufficiently large regular cardinal. Then, because κ is Mahlo, we can take M such that

- $\kappa, \mathbb{R}(\kappa), \dot{B}, \dot{C}, q \in M \prec \langle \mathcal{H}_\theta, \in \rangle$,
- $M \cap \kappa = |M| =: \mu$ is an inaccessible cardinal.

Let $\langle D_\alpha \mid \alpha < \mu \rangle$ be an enumeration of all dense open subsets of $\mathbb{R}(\kappa)$ which belong to M.

By induction on $\alpha < \mu$ construct a descending sequence $\langle p_\alpha \mid \alpha < \mu \rangle$ in $\mathbb{R}(\kappa) \cap M$ as follows:

Let $p_0 := q$. If α is a successor ordinal, then let $p_\alpha \leq p_{\alpha-1}$ be such that $p_\alpha \in D_{\alpha-1} \cap M$ and such that $p(b) = 0$ for all $b \in {}^{\upsilon_{p_\alpha}}2$. Suppose that α is a limit ordinal and that $\langle p_\beta \mid \beta < \alpha \rangle$ has been taken. Let $\upsilon := \sup_{\beta<\alpha} \upsilon_{p_\beta}$, and let p_α be a function on ${}^{\leq \upsilon}2$ such that $p_\alpha \restriction {}^{<\upsilon}2 = \bigcup_{\beta<\alpha} p_\beta$ and such that $p(b) = 0$ for all $b \in {}^{\upsilon}2$.

Note that if α is a limit ordinal, then $C := \{\upsilon_{p_\beta} \mid \beta < \alpha\}$ is club in υ_{p_α}, and $p_\alpha(b) = 0$ for all b with $\mathrm{dom}(b) \in C$. Then, using Lem.4.4 (2), by induction on α we can easily show that $p_\alpha \in \mathbb{R}(\kappa) \cap M$.

Now we have constructed $\langle p_\alpha \mid \alpha < \mu \rangle$. Then by its genericity, any lower bound of $\{p_\alpha \mid \alpha < \mu\}$ forces that $\alpha \in \dot{C}$. Moreover we can take $b^* \in {}^{\mu}2$ such that any lower bound of $\{p_\alpha \mid \alpha < \mu\}$ forces that $b^* = \dot{B} \restriction \mu$.

Let p be a function on ${}^{\leq \mu}2$ such that $p \restriction {}^{<\mu}2 = \bigcup_{\alpha<\mu} p_\alpha$, such that $p(b^*) = 1$ and such that $p(b) = 0$ for any $b \in {}^{\mu}2 \setminus \{b^*\}$. Then it is easy to see that p and μ are as desired. $\square$

Lemma 4.9. *Suppose that κ is an inaccessible cardinal. Then κ is not Π^1_1-indescribable in $V^{\mathbb{R}(\kappa)}$.*

Proof. First note that if κ is not Mahlo in V, then so is in $V^{\mathbb{R}(\kappa)}$, and thus κ is not Π^1_1-indescribable. Hence we may assume that κ is Mahlo in V.

Let H be an $\mathbb{R}(\kappa)$-generic filter over V, and let $F := \bigcup H$. By Lem.4.8 for any $B \in {}^{\kappa}2$ the set $\{\mu < \kappa \mid F(B \restriction \mu) = 1\}$ is stationary in κ. But for each $\lambda < \kappa$ there is $B \in {}^{\lambda}2$ such that the set $\{\mu < \lambda \mid F(B \restriction \mu) = 1\}$ is non-stationary in λ by the definition of $\mathbb{R}(\kappa)$. Therefore κ is not Π^1_1-indescribable. $\square$

Lemma 4.10. *Suppose that κ be a Π^1_1-indescribable cardinal. Moreover let $\langle \mathbb{P}_\mu, \dot{\mathbb{Q}}_\nu \mid \nu \leq \kappa, \mu \leq \kappa + 1 \rangle$ be the reverse Easton support iteration of length $\kappa + 1$ defined as follows:*

- *If ν is an inaccessible cardinal $\leq \kappa$, then $\dot{\mathbb{Q}}_\nu$ is a $\mathbb{P}_\nu$-name for the poset $\mathbb{R}(\nu) * \mathbb{C}^*(\bigcup \dot{H}_\nu)$, where $\dot{H}_\nu$ is the canonical name for a $\mathbb{R}(\nu)$-generic filter.*
- *If ν is not inaccessible, then $\dot{\mathbb{Q}}_\nu$ is a $\mathbb{P}_\nu$-name for a trivial poset.*

Then κ is Π^1_1-indescribable in $V^{\mathbb{P}_{\kappa+1}}$.

Proof. Suppose that G is $\mathbb{P}_{\kappa+1}$-generic filter over V. In $V[G]$ suppose that $A \subseteq V_\kappa$, that $\varphi(X)$ is a Π^1_1-formula and that $V_\kappa \models \varphi[A]$. We show that in $V[G]$ there exists $\lambda < \kappa$ with $V_\lambda \models \varphi[A \cap V_\lambda]$.

First we make some preliminaries in V. In V let $\dot{A}$ be a $\mathbb{P}_{\kappa+1}$-name of A. We may assume that $\dot{A} \in \mathcal{H}_{\kappa^+}$ in V. In V let θ be a sufficiently large regular cardinal, and take an $M \prec \langle \mathcal{H}_\theta, \in \rangle$ such that $|M| = \kappa \subseteq M$, such that ${}^{<\kappa}M \subseteq M$ and such that $\mathbb{P}_{\kappa+1}, \dot{A} \in M$. Let $\sigma : M \to N$ be the transitive collapse. Because κ is Π^1_1-indescribable, in V we can take a transitive model K of ZFC^- and an elementary embedding $j : N \to K$ such that ${}^{<\kappa}N \subseteq K$ and such that the critical point of j is κ.

Let $\mathbb{P}' := \mathbb{P}_{\kappa+1} \cap N = \sigma(\mathbb{P}_{\kappa+1})$, and let $G' := G \cap \mathbb{P}'$. Then in $V[G]$, by the standard master condition argument, we can take a $j(\mathbb{P}')$-generic filter $\bar{G}'$ over K and extend j to an elementary embedding $j : N[G'] \to K[\bar{G}']$. Then in $K[\bar{G}']$ it holds that $V_\kappa \models \varphi[A]$. So it holds in $K[\bar{G}']$ that there exists $\lambda < j(\kappa)$ with $V_\lambda \models \varphi[j(A) \cap V_\lambda]$. Hence by the elementarity of j it holds in $N[G']$ that there exists $\lambda < \kappa$ with $V_\lambda \models \varphi[A \cap V_\lambda]$. Here note that $(V_\kappa)^{V[G]} \subseteq M[G']$. Hence it holds in $V[G]$ that there is $\lambda < \kappa$ with $V_\lambda \models \varphi[A \cap V_\lambda]$. □

Now we can easily prove Thm.4.1:

Proof of Thm.4.1. Let $\langle \mathbb{P}_\mu, \dot{\mathbb{Q}}_\nu \mid \mu \leq \kappa, \nu < \kappa \rangle$ be the reverse Easton support iteration defined as follows:

- If ν is an inaccessible cardinal $< \kappa$, then $\dot{\mathbb{Q}}_\nu$ is a $\mathbb{P}_\nu$-name for the poset $\mathbb{R}(\nu) * \mathbb{C}^*(\bigcup \dot{H}_\nu)$, where $\dot{H}_\nu$ is the canonical name for a $\mathbb{R}(\nu)$-generic filter.
- If ν is not inaccessible, then $\dot{\mathbb{Q}}_\nu$ is a $\mathbb{P}_\nu$-name for a trivial poset.

Let $\mathbb{P} := \mathbb{P}_\kappa * \dot{\mathbb{R}}(\kappa)$. We claim that $V^{\mathbb{P}}$ is an forcing extension witnessing the theorem. Suppose that $G = G_\kappa * H_\kappa$ is a $\mathbb{P}$-generic filter over V, where G_κ is a $\mathbb{P}_\kappa$-generic filter over V, and H_κ is a $\mathbb{R}(\kappa)$-generic filter over $V[G_\kappa]$.

First note that κ is inaccessible in $V[G]$ by Lem.4.4 (1) and 4.7. Moreover κ is not Π^1_1-indescribable in $V[G]$ by Lem.4.9. Thus the property (I) of Thm.4.1 holds in $V[G]$.

To check the property (II) of Thm.4.1, in $V[G]$ take an arbitrary sequence $\langle S_\alpha \mid \alpha < \kappa \rangle$ of stationary subsets of κ. We must find an inaccessible $\lambda < \kappa$ such that $S_\alpha \cap \lambda$ is stationary for all $\alpha < \lambda$. For this take a $\mathbb{C}^*(\bigcup H_\kappa)$-generic filter I_κ over $V[G]$. Then κ is Π^1_1-indescribable in $V[G * I_\kappa]$ by Lem.4.10. Moreover each S_α remains stationary in $V[G * I_\kappa]$. Hence in $V[G * I_\kappa]$ we can take an inaccessible $\lambda < \kappa$ such that $S_\alpha \cap \lambda$ is stationary for all $\alpha < \lambda$. Then, also in $V[G]$, λ is an inaccessible cardinal, and $S_\alpha \cap \lambda$ is stationary for all $\alpha < \lambda$. □

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