# Ideals over $\omega_2$ and combinatorical principles on $\omega_1$

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**Abstract**

An ideal $I$ over $\mathcal{P}_\kappa\lambda$ is said to be $\sigma$-strategically closed, proper or semiproper if the corresponding poset $\mathbb{P}_I$ is $\sigma$-strategically closed, proper or semiproper, respectively. In this note we prove the following:

- If there exists a $\sigma$-strategically closed, fine and $\kappa$-complete ideal over $\mathcal{P}_{\omega_2}\lambda$ for some $\lambda \geq \omega_2$ then $\diamondsuit_{\omega_1}$ holds.
- If there exists a proper, fine and $\kappa$-complete ideal over $\mathcal{P}_{\omega_2}\lambda$ for some $\lambda \geq \omega_2$ then $\mathsf{CB}$, the completely bounding principle, does not hold.
- If there exists a semiproper, fine and $\kappa$-complete ideal over $\mathcal{P}_{\omega_2}\lambda$ for some $\lambda \geq \omega_2$ then $\diamondsuit^*_{\omega_1}$ does not hold.

## 1 Introduction

An ideal $I$ over $\mathcal{P}_\kappa\lambda$ is said to be $\sigma$-strategically closed, proper or semiproper if the corresponding poset $\mathbb{P}_I$ is $\sigma$-strategically closed, proper or semiproper, respectively. (In usual $I$ is said to be proper if whole $\mathcal{P}_\kappa\lambda$ is not in $I$. In this note this meaning of properness is included in being an ideal. See Section 2.) We prove the following:

**Theorem 1.1.** *Assume that there exists a $\sigma$-strategically closed, fine and $\omega_2$-complete ideal over $\mathcal{P}_{\omega_2}\lambda$ for some $\lambda \geq \omega_2$. Then $\diamondsuit_{\omega_1}$ holds.*

**Theorem 1.2.** *Assume that there exists a proper, fine and $\omega_2$-complete ideal over $\mathcal{P}_{\omega_2}\lambda$ for some $\lambda \geq \omega_2$. Then* $\mathsf{CB}$*, the completely bounding principle, does not hold.*

**Theorem 1.3.** *Assume that there exists a semiproper, fine and $\omega_2$-complete ideal over $\mathcal{P}_{\omega_2}\lambda$ for some $\lambda \geq \omega_2$. Then $\diamondsuit^*_{\omega_1}$ does not hold.*

By Theorem 1.2 the existence of a proper ideal over $\mathcal{P}_{\omega_2}\lambda$ implies that $\mathrm{NS}_{\omega_1}$ is not $\omega_2$-saturated. On the other hand, it is shown in Sakai [3] that the existence of a semiproper ideal over $\mathcal{P}_{\omega_2}\lambda$ and Martin's maximum are consistent. Thus the existence of a semiproper ideal over $\mathcal{P}_{\omega_2}\lambda$ does not imply that $\mathrm{NS}_{\omega_1}$ is not $\omega_2$-saturated.

# 2 Preliminaries

Here we give our notations and facts about posets, ideals and combinatorical principles.

**Posets**: For the definition and basics of properness and semiproperness of posets, consult Jech [2]. The only fact that we use explicitly is that if $\mathbb{P}$ is proper then $\mathbb{P}$ preserves stationary subsets of $\mathcal{P}_{\omega_1}\gamma$ for every $\gamma \geq \omega_1$.

Here we present the definition of $\sigma$-strategically closure of posets. For a poset $\mathbb{P}$ let $\Game(\mathbb{P})$ be the following two players' game of GOOD and BAD of length $\omega$:

| BAD | $p_0$ | $p_1$ | $p_2$ | $\cdots$ | $p_n$ | $\cdots$ |
|---|---|---|---|---|---|---|
| GOOD | $q_0$ | $q_1$ | $q_2$ | $\cdots$ | $q_n$ | $\cdots$ |

BAD and GOOD in turn choose elements of $\mathbb{P}$ and construct a descending sequence in $\mathbb{P}$. In the 0-th stage, first BAD opens a game by choosing an arbitrary $p_0 \in \mathbb{P}$, and then GOOD plays a $q_0 \leq p_0$. In the $n$-th stage for $n > 0$, first BAD plays $p_n \leq q_{n-1}$ and then GOOD plays $q_n \leq p_n$. GOOD wins if $\{q_n \mid n \in \omega\}$ has a lower bound in $\mathbb{P}$. Otherwise BAD wins.

A function $\tau : {}^{<\omega}\mathbb{P} \to \mathbb{P}$ is called a winning strategy for GOOD in $\Game(\mathbb{P})$ if GOOD wins in $\Game(\mathbb{P})$ whenever he plays $\tau(\langle p_m \mid m \leq n\rangle)$ in each $n$-th stage. $\mathbb{P}$ is said to be *$\sigma$-strategically closed* if there exists a winning strategy for GOOD in the game $\Game(\mathbb{P})$.

**Ideals**: Let $\kappa$ be a regular uncountable cardinal, and let $\lambda$ be an ordinal with $\lambda \geq \kappa$.

A family $I$ of subsets of $\mathcal{P}_\kappa\lambda$ is called a *$\kappa$-ideal over $\mathcal{P}_\kappa\lambda$* if $I$ is a fine $\kappa$-complete ideal over $\mathcal{P}_\kappa\lambda$ with $\mathcal{P}_\kappa\lambda \notin I$, that is, $I$ satisfies the following:

(1) $\bigcup \mathcal{X} \in I$ for every $\mathcal{X} \subseteq I$ with $|\mathcal{X}| < \kappa$. ($\kappa$-completeness)

(2) $X \in I \wedge Y \subseteq X \ \to \ Y \in I$ for every $X, Y \subseteq \mathcal{P}_\kappa\lambda$. (downward closure)

(3) $\{\gamma\} \in I$ for every $\gamma \in \lambda$. (fineness)

(4) $\mathcal{P}_\kappa\lambda \notin I$. (properness in the usual meaning)

Here note that properness in the usual meaning is included in being a $\kappa$-ideal.

Let $I$ be a $\kappa$-ideal over $\mathcal{P}_\kappa\lambda$. Then let $\mathbb{P}_I$ denotes the poset of all $I$-positive sets ordered by inclusions. That is $\mathbb{P}_I = \langle \mathcal{P}(\mathcal{P}_\kappa\lambda) \setminus I\,,\subseteq\rangle$. $I$ is said to be *$\sigma$-strategically closed*, *proper* or *semiproper* if $\mathbb{P}_I$ is $\sigma$-strategically closed, proper or semiproper, respectively.

We use the following fact:

**Fact 2.1** (Gitik-Shelah [1])**.** *Let $\kappa$ be a regular uncountable cardinal and let $\lambda$ be an ordinal with $\lambda \geq \kappa$. If $I$ is a proper $\kappa$-ideal over $\mathcal{P}_\kappa\lambda$ then $I$ is precipitous.*

**Combinatorical principles**: $\diamondsuit_{\omega_1}$, $\diamondsuit^*_{\omega_1}$ and CB are the following principles:

$\diamondsuit_{\omega_1} \ \equiv$ There exists a sequence $\langle f_\alpha \mid \alpha \in \omega_1\rangle$ such that

(i) $f_\alpha \in {}^{\alpha}2$ for each $\alpha \in \omega_1$,

(ii) for every $F \in {}^{\omega_1}2$ the set $\{\alpha \in \omega_1 \mid F \restriction \alpha = f_\alpha\}$ is stationary in $\omega_1$.

$\diamondsuit^*_{\omega_1}$ $\equiv$ There exists a sequence $\langle \mathcal{F}_\alpha \mid \alpha \in \omega_1 \rangle$ such that

(i) $\mathcal{F}_\alpha$ is a countable subset of ${}^{\alpha}2$ for each $\alpha \in \omega_1$,

(ii) for every $F \subseteq {}^{\omega_1}2$ the set $\{\alpha \in \omega_1 \mid F \restriction \alpha \in \mathcal{F}_\alpha\}$ contains a club in $\omega_1$.

$\mathsf{CB}$ $\equiv$ For every function $H : \omega_1 \to \omega_1$ there exists $\gamma \in [\omega_1, \omega_2)$ and a bijection $\pi : \omega_1 \to \gamma$ such that the set $\{\alpha \in \omega_1 \mid H(\alpha) < \text{o.t.}(\pi``\alpha)\}$ contains a club in $\omega_1$.

A sequence $\langle f_\alpha \mid \alpha \in \omega_1 \rangle$ witnessing $\diamondsuit_{\omega_1}$ is called a $\diamondsuit_{\omega_1}$-sequence, and a sequence $\langle \mathcal{F}_\alpha \in \omega_1 \rangle$ witnessing $\diamondsuit^*_{\omega_1}$ is called a $\diamondsuit^*_{\omega_1}$-sequence.

In the proof of Theorem 1.2 we use a diamond principle on $\mathcal{P}_{\omega_1}\lambda$. For an ordinal $\lambda \geq \omega_1$ let $\diamondsuit_{\omega_1}\lambda$ be the following principle:

$\diamondsuit_{\omega_1\lambda}$ $\equiv$ There exists a sequence $\langle f_x \mid x \in \mathcal{P}_{\omega_1}\lambda \rangle$ such that

(i) $f_x \in {}^{x}2$ for each $x \in \mathcal{P}_{\omega_1}\lambda$,

(ii) for every $F \subseteq {}^{\lambda}2$ the set $\{x \in \mathcal{P}_{\omega_1}\lambda \mid F \restriction x = f_x\}$ is stationary in $\mathcal{P}_{\omega_1}\lambda$.

We call a sequence $\langle f_x \mid x \in \mathcal{P}_{\omega_1}\lambda \rangle$ witnessing $\diamondsuit_{\omega_1\lambda}$ a $\diamondsuit_{\omega_1\lambda}$-sequence.

Note that $\diamondsuit_{\omega_1\omega_1}$ is equivalent to $\diamondsuit_{\omega_1}$. Hence it is independent of $\mathsf{ZFC}$. On the other hand, $\diamondsuit_{\omega_1\lambda}$ is a consequence of $\mathsf{ZFC}$ for $\lambda \geq \omega_2$:

**Fact 2.2** (Shelah)**.** *$\diamondsuit_{\omega_1\lambda}$ holds for every $\lambda \geq \omega_2$.*

This fact is a key of the proof of Theorem 1.2. The proof of this fact can be found in Shioya [4].

# 3 $\sigma$-strategically closed ideal and $\diamondsuit_{\omega_1}$

Here we prove Theorem 1.1. A key of our proof is the following theorem due to Baumgartner:

**Theorem 3.1** (Baumgartner)**.** *Suppose that $\mathbb{P}$ is a $\sigma$-strategically closed poset which adds new subsets of $\omega_1$. Then $\Vdash_{\mathbb{P}} \diamondsuit_{\omega_1}$.*

Theorem 1.1 is an easy corollary of the proof of Theorem 3.1. Outline of the proof of Theorem 1.1 is as follows:

Suppose that $I$ is a $\sigma$-strategically closed ideal over $\mathcal{P}_{\omega_2}\lambda$ for some $\lambda \geq \omega_2$. Note that $I$ is precipitous by Fact 2.1. Take a $\mathbb{P}_I$-generic filter $G$ over $V$, let $M$ be the transitive collapse of the ultrapower of $V$ by $G$, and let $j : V \to M$ be the ultrapower map. Then the critical point of $j$ is $\omega_2^V$. In particular

$\mathcal{P}(\omega_1)^V \subsetneq \mathcal{P}(\omega_1)^M \subseteq \mathcal{P}(\omega_1)^{V[G]}$. Hence $\diamondsuit_{\omega_1}$ holds in $V[G]$ by Theorem 3.1. In fact, by a close investigation of the proof of Theorem 3.1, it turns out that a $\diamondsuit_{\omega_1}$-sequence can be taken in $M$. Thus $\diamondsuit_{\omega_1}$ holds in $M$. Then $\diamondsuit_{\omega_1}$ holds in $V$ by the elementarity of $j$.

First we give the proof of Theorem 3.1. Before starting, we prove the following:

**Lemma 3.2.** *For every countable limit ordinal $\alpha$ there exists a map $\pi : {}^{\alpha}2 \to {}^{\alpha}2$ with the following property:*

*For every sequence $\langle h_s, f_s \mid s \in {}^{<\omega}2\rangle$ such that*

(1) $s \subseteq t \ \Leftrightarrow\ h_s \subseteq h_t$ *for each* $s, t \in {}^{<\omega}2$,

(2) $s \subseteq t \ \Rightarrow\ f_s \subseteq f_t$ *for each* $s, t \in {}^{<\omega}2$,

(3) $\sup_{n\in\omega}(\mathrm{dom}\, h_{b\restriction n}) = \sup_{n\in\omega}(\mathrm{dom}\, f_{b\restriction n}) = \alpha$ *for every* $b \in {}^{\omega}2$,

*there exists $b \in {}^{\omega}2$ with $\pi(\bigcup_{n\in\omega} h_{b\restriction n}) = \bigcup_{n\in\omega} f_{b\restriction n}$.*

*Proof.* First note that there are at most $2^\omega$-many sequences $\langle h_s, f_s \mid s \in {}^{<\omega}2\rangle$ with the properties (1)-(3). Let $\langle \vec{z}_\xi \mid \xi < 2^\omega\rangle$ be an enumeration of such sequences, and let $\vec{z}_\xi$ be $\langle h^\xi_s, f^\xi_s \mid s \in {}^{<\omega}2\rangle$.

Note also that the map $b \mapsto \bigcup_{n\in\omega} h^\xi_{b\restriction n}$ is injective for each $\xi < 2^\omega$. Hence we can inductively construct a seqeunce $\langle b_\xi \mid \xi < 2^\omega\rangle$ in ${}^{\omega}2$ so that $\langle \bigcup_{n\in\omega} h^\xi_{b_\xi\restriction n} \mid \xi < 2^\omega\rangle$ is pairwise distinct.

Then we can take a map $\pi : {}^{\omega}2 \to \omega 2$ such that

$$\pi\Big(\bigcup_{n\in\omega} h^\xi_{b_\xi\restriction n}\Big) \;=\; \bigcup_{n\in\omega} f^\xi_{b_\xi\restriction n} \,.$$

This $\pi$ witnesses the lemma. □

Now we give the proof of Theorem 3.1:

*Proof of Thm.3.1.* For each countable limit ordinal $\alpha$ take a map $\pi_\alpha : {}^{\alpha}2 \to {}^{\alpha}2$ witnessing Lemma 3.2, and for each countable successor ordinal $\alpha$ let $\pi_\alpha$ be the identity map on ${}^{\alpha}2$. Also, take a $\mathbb{P}$-name $\dot{H}$ of a function from $\omega_1$ to 2 which does not belong to $V$. We show that

$$\Vdash_{\mathbb{P}} \text{“} \langle \pi_\alpha(\dot{H} \restriction \alpha) \mid \alpha \in \omega_1\rangle \text{ is a } \diamondsuit_{\omega_1}\text{-sequence”} \,.$$

Here note that $\Vdash_{\mathbb{P}}$ “$\dot{H} \restriction \alpha \in V$” for each countable $\alpha$ because $\mathbb{P}$ is $\sigma$-strategically closed.

Take an arbitrary $p \in \mathbb{P}$, an arbitrary $\mathbb{P}$-name of a function from $\omega_1$ to 2 and an arbitrary $\mathbb{P}$-name $\dot{C}$ of a club subset of $\omega_1$. It suffices to find $q \leq p$ and $\alpha \in \omega_1$ such that $q \Vdash_{\mathbb{P}}$ “$\alpha \in \dot{C} \ \wedge\ \dot{F} \restriction \alpha = \pi_\alpha(\dot{H} \restriction \alpha)$”.

Let $\tau$ be a winning strategy for `GOOD` in the game $\eth(\mathbb{P})$. Let $\theta$ be a sufficiently large regular cardinal, and take a countable elementary submodel $M$ of $\langle \mathcal{H}_\theta, \in\rangle$

with $\mathbb{P}, \dot{H}, p, \dot{F}, \dot{C} \in M$. Let $\alpha$ be $M \cap \omega_1$. Also, take an increasing cofinal sequence $\langle \beta_n \mid n \in \omega \rangle$ in $\alpha$.

By induction on the length of $s \in {}^{<\omega}2$ take $p_s, q_s \in \mathbb{P} \cap M$ and $h_s, f_s \in {}^{<\alpha}2 \cap M$ as follows. The induction hypothesis is that $p_s \Vdash_{\mathbb{P}}$ "$h_s \subseteq \dot{H} \wedge f_s \subseteq \dot{F}$".

First let $p_\emptyset := p$, $q_\emptyset := \tau(\langle p \rangle)$ and let $h_\emptyset = f_\emptyset := \emptyset$.

Suppose that $p_s$, $q_s$, $h_s$ and $f_s$ have been defined. Then we can take $p_{s^\frown\langle i \rangle} \in \mathbb{P} \cap M$ and $h_{s^\frown\langle i \rangle}, f_{s^\frown\langle i \rangle} \in {}^{<\alpha}2 \cap M$ for $i \in 2$ so that

(I) $h_s \subseteq h_{s^\frown\langle i \rangle} \ \wedge \ f_s \subseteq f_{s^\frown\langle i \rangle}$,

(II) $\beta_{|s|} \leq \operatorname{dom} h_{s^\frown\langle i \rangle}, \operatorname{dom} f_{s^\frown\langle i \rangle}$,

(III) $p_{s^\frown\langle i \rangle} \Vdash_{\mathbb{P}}$ "$h_s \subseteq \dot{H} \ \wedge \ f_s \subseteq \dot{F}$",

(IV) $h_{s^\frown\langle 0 \rangle} \not\subseteq h_{s^\frown\langle 1 \rangle} \ \wedge \ h_{s^\frown\langle 1 \rangle} \not\subseteq h_{s^\frown\langle 0 \rangle}$.

(III) and (IV) can be realized because $\dot{H}$ is forced not to belong to $V$. Finally let $q_s := \tau(\langle p_{s\restriction m} \mid m \leq |s| \rangle)$. This completes the induction.

Now $\langle h_s, f_s \mid s \in {}^{<\omega}2 \rangle$ satisfies the properties (1)-(3) in Lemma 3.2. Hence we can take $b \in {}^{\omega}2$ with

$$\pi_\alpha\Big( \bigcup_{n \in \omega} h_{b\restriction n} \Big) \ = \ \bigcup_{n \in \omega} f_{b\restriction n} \ .$$

Note that $\langle p_{b\restriction n}, q_{b\restriction n} \mid n \in \omega \rangle$ is a sequence of moves in $\Game(\mathbb{P})$ in which `GOOD` has played according to a winning strategy $\tau$. Hence we can take a lower bound $q$ of $\{p_{b\restriction n} \mid n \in \omega\}$. Then

$$q \Vdash_{\mathbb{P}} \text{“} \dot{H} \restriction \alpha = \bigcup_{n \in \omega} h_{b\restriction n} \ \wedge \ \dot{F} \restriction \alpha = \bigcup_{n \in \omega} f_{b\restriction n}$$

by the construction of $\langle p_s, h_s, f_s \mid s \in {}^{<\omega}2 \rangle$. Thus

$$q \Vdash_{\mathbb{P}} \text{“} \dot{F} \restriction \alpha \ = \ \pi_\alpha(\dot{H} \restriction \alpha) \text{”} \ .$$

Recall that $p_0 = p$. Thus $q \leq p$. Therefore $q$ and $\alpha$ are ones we desired.

This completes the proof. □

Using the above proof, we can prove Theorem 1.1:

*Proof of Thm.1.1.* Suppose that $\lambda \geq \omega_2$ and that $I$ is a $\sigma$-strategically closed $\omega_2$-ideal over $\mathcal{P}_{\omega_2}\lambda$. Note that $I$ is precipitous by Fact 2.1. Take a $\mathbb{P}_I$-generic filter $G$ over $V$, let $M$ be the transitive collapse of the ultrapower of $V$ by $G$, and let $j : V \to M$ be the ultrapower map.

First note that $|\omega_2^V| = \omega_1$ in $M$ because the critical point of $j$ is $\omega_2^V$. Hence we can take $F \in {}^{\omega_1}2 \cap M$ which is not in $V$.

Next, in $V$, for each countable limit ordinal $\alpha$ take a map $\pi_\alpha : {}^{\alpha}2 \to {}^{\alpha}2$ witnessing Lemma 3.2, and for each countable successor ordinal $\alpha$ let $\pi_\alpha$ be the identity map on ${}^{\alpha}2$. Here note that for each $\alpha \in \omega_1$, ${}^{\alpha}2$ is absolute among $V$,

$V[G]$ and $M$ because $\mathbb{P}_I$ is $\sigma$-strategically closed. Hence $j(\pi_\alpha) = \pi_\alpha$ for each $\alpha \in \omega_1$. Thus $\langle \pi_\alpha \mid \alpha \in \omega_1 \rangle = j(\langle \pi_\alpha \mid \alpha \in \omega_1 \rangle) \in M$.

Now $\langle \pi_\alpha(H \restriction \alpha) \mid \alpha \in \omega_1 \rangle$ is in $M$. Moreover $\langle \pi_\alpha(H \restriction \alpha) \mid \alpha \in \omega_1 \rangle$ is a $\diamondsuit_{\omega_1}$-sequence in $V[G]$ by the proof of Theorem 3.1, and thus so in $M$. Therefore $\diamondsuit_{\omega_1}$ holds in $M$. Then $\diamondsuit_{\omega_1}$ holds in $V$ by the elementarity of $j$. □

# 4 Proper ideal and CB

Here we prove Theorem 1.2. Theorem 1.2 is a corollayr of (the proof of) the following Theorem:

**Theorem 4.1.** *Suppose that $\mathbb{P}$ is a proper poset which collapses $\omega_2$, that is, $\Vdash_{\mathbb{P}}$ "$|\omega_2^V| = \omega_1$ ". Then $\Vdash_{\mathbb{P}} \neg$CB.*

The proof of Theorem 1.2 is as follows. Suppose that $I$ is a proper $\omega_2$-ideal over $\mathcal{P}_{\omega_2}\lambda$. Take a $\mathbb{P}_I$-generic filter $G$ over $V$, let $M$ be the transitive collapse of the ultrapower of $V$ by $G$, and let $j : V \to M$ the ultrapower map. Then $\omega_2^V < \omega_2^M \leq \omega_2^{V[G]}$ because the critical point of $j$ is $\omega_2^V$. Hence CB does not hold in $V[G]$ by Theorem 4.1. In fact the close observation of the proof of Theorem 4.1 proves that CB does not hold in $M$. Then CB does not hold in $V$ by the elementarity of $j$.

Prior to the proof of Theorem 4.1, we reformulate CB:

**Lemma 4.2.** *The following are equivalent:*

(1) CB

(2) *For every function $H : \omega_1 \to \omega_1$ there exists $\gamma \in [\omega_1, \omega_2)$ such that the set $\{x \in \mathcal{P}_{\omega_1}\gamma \mid x \cap \omega_1 \in \omega_1 \ \wedge \ H(x \cap \omega_1) < \text{o.t.}\, x\}$ contains a club in $\mathcal{P}_{\omega_1}\gamma$.*

(3) *For every $\delta \in [\omega_1, \omega_2)$ and every function $H : \mathcal{P}_{\omega_1}\delta \to \omega_1$, there exists $\gamma \in [\delta, \omega_2)$ such that the set $\{x \in \mathcal{P}_{\omega_1}\gamma \mid H(x \cap \delta) < \text{o.t.}\, x\}$ contains a club in $\mathcal{P}_{\omega_1}\gamma$.*

*Proof.* **(1)⇒(2)**: Suppose that CB holds. To show that (2) holds, take an arbitrary fucntion $H : \omega_1 \to \omega_1$. By CB there exists $\gamma \in [\omega_1, \omega_2)$ and a bijection $\pi : \omega_1 \to \gamma$ such that the set $C := \{\alpha \in \omega_1 \mid H(\alpha) < \text{o.t.}(\pi``\alpha)\}$ contains a club in $\omega_1$. Let $C^*$ be the set of all $x \in \mathcal{P}_{\omega_1}\gamma$ such that $x \cap \omega_1 \in C^*$ and $x$ is closed under $\pi$ and $\pi^{-1}$. Then $C^*$ contains a club in $\mathcal{P}_{\omega_1}\gamma$.

It suffices to show that if $x \in C^*$ then $H(x \cap \omega_1) < \text{o.t.}(x)$. Suppose that $x \in C^*$. Then $x = \pi``(x \cap \omega_1)$ because $x$ is closed under $\pi$ and $\pi^{-1}$. Then, because $x \cap \omega_1 \in C$, $H(x \cap \omega_1) < \text{o.t.}\, \pi``(x \cap \omega_1) = \text{o.t.}\, x$.

**(2)⇒(3)**: Assume that (2) holds. To show that (3) holds, take an arbitrary $\delta \in [\omega_1, \omega_2)$ and an arbitrary function $H : \mathcal{P}_{\omega_1}\delta \to \omega_1$. Take a bijection $\tau : \omega_1 \to \delta$ and let $F : \omega_1 \to \omega_1$ be the function defined as $F(\alpha) := H(\tau``\alpha)$ for each $\alpha \in \omega_1$. Then, by (2) there exists $\gamma \in [\omega_1, \omega_2)$ such that $C := \{x \in \mathcal{P}_{\omega_1}\gamma \mid x \cap \omega_1 \in \omega_1 \ \wedge \ F(x \cap \omega_1) < \text{o.t.}\, x\}$ contains a club.

Note that if $\gamma \leq \gamma' < \omega_2$ then the set $\{x \in \mathcal{P}_{\omega_1}\gamma' \mid x \cap \omega_1 \in \omega_1 \wedge F(x \cap \omega_1) < \text{o.t.}\, x\}$ contains a club. This is because the set $\{x \in \mathcal{P}_{\omega_1}\gamma' \mid x \cap \gamma \in C\}$ contains a club and $F(x \cap \omega_1) < \text{o.t.}(x \cap \gamma) \leq \text{o.t.}\, x$ for every $x \in \mathcal{P}_{\omega_1}\gamma'$ with $x \cap \gamma \in C$. Hence we may assume that $\gamma \geq \delta$.

Then $C^* := \{x \in C \mid x \text{ is closed under } \tau, \tau^{-1}\}$ contains a club in $\mathcal{P}_{\omega_1}\gamma$. Note that $x \cap \delta = \tau``(x \cap \omega_1)$ for every $x \in C^*$. Hence for every $x \in C^*$,

$$H(x \cap \delta) \ = \ H(\tau``(x \cap \omega_1)) \ = \ F(x \cap \omega_1) \ < \ \text{o.t.}\, x \ .$$

Therefore (3) holds.

**(3)⇒(1)**: Assume that (3) holds. Take an arbitrary function $H : \omega_1 \to \omega_1$. Then, by (3) there exists $\gamma \in [\omega_1, \omega_2)$ such that $C := \{x \in \mathcal{P}_{\omega_1}\gamma \mid x \cap \omega_1 \in \omega_1 \ \wedge \ H(x \cap \omega_1) < \text{o.t.}(x)\}$.

Take a bijection $\pi : \omega_1 \to \gamma$ and let $C^* := \{\alpha \in \omega_1 \mid \pi``\alpha \in C \wedge (\pi``\alpha) \cap \omega_1 = \alpha\}$. Then $C^*$ contains a club in $\omega_1$. Moreover for each $\alpha \in C^*$,

$$H(\alpha) \ = \ H((\pi``\alpha) \cap \omega_1) \ < \ \text{o.t.}(\pi``\alpha) \ .$$

Therefore (1) holds. □

Now we prove the Theorem 4.1:

*Proof of Thm.4.1.* By Fact 2.2 we can take a sequence $\langle b_z \mid z \in \mathcal{P}_{\omega_1}\omega_2 \rangle$ with the following properties:

(i) $b_z \subseteq z \times z$ for every $z \in \mathcal{P}_{\omega_1}\omega_2$.

(ii) For every $B \subseteq \omega_2 \times \omega_2$ the set $\{z \in \mathcal{P}_{\omega_1}\omega_2 \mid B \cap (z \times z) = b_z\}$ is stationary in $\mathcal{P}_{\omega_1}\omega_2$.

Define a function $H : \mathcal{P}_{\omega_1}\omega_2 \to \omega_1$ as follows: If $b_z$ is a well-ordering of $z$ then let $H(z)$ be $\ln(b_z)$, the length of the well-ordering $b_z$. Otherwise let $H(z)$ be 0.

Recall that $\mathbb{P}$ collapses $\omega_2$. Hence, by Lemma 4.2, it suffices to show that

$$\Vdash_{\mathbb{P}} \text{“}\{x \in \mathcal{P}_{\omega_1}\gamma \mid H(x \cap \omega_2^V) \geq \text{o.t.}\, x\} \text{ is stationary in } \mathcal{P}_{\omega_1}\gamma\text{”}$$

for every $\gamma \geq \omega_2$. But $\mathbb{P}$ is proper. Thus the following claim suffices for this:

**Claim .** *For every $\gamma \geq \omega_2$ the set $\{x \in \mathcal{P}_{\omega_1}\gamma \mid H(x \cap \omega_2) \geq \text{o.t.}\, x\}$ is stationary in $\mathcal{P}_{\omega_1}\gamma$.*

⊢ It suffices to show that the following:

($\star$) For every $X \subseteq \gamma$ with $|X| = \omega_2 \subseteq X$, the set $\{x \in \mathcal{P}_{\omega_1}X \mid H(x \cap \omega_2) \geq \text{o.t.}\, x\}$ is stationary in $\mathcal{P}_{\omega_1}X$.

Suppose that $X \subseteq \gamma$ and $|X| = \omega_2 \subseteq X$. Take a bijection $\pi : \omega_2 \to X$. Let $B$ be the well-ordering of $\omega_2$ induced from $\pi$, that is, let $B := \{\langle \xi, \eta \rangle \mid \pi(\xi) < \pi(\eta)\}$. Moreover let $S$ be the set of all $x \in \mathcal{P}_{\omega_1}\gamma$ such that

(I) $x$ is closed under $\pi$ and $\pi^{-1}$,

(II) $B \cap (x \cap \omega_2 \times x \cap \omega_2) = b_{x \cap \omega_2}$.

Then $S$ is staionary by the choice of $\langle b_z \mid z \in \mathcal{P}_{\omega_1}\omega_2 \rangle$. Note that $x = \pi``(x \cap \omega_2)$ for each $x \in S$. Hence, by (II) and the construction of $H$,

$$\begin{aligned} \text{o.t.}\, x &= \text{o.t.}(\pi``(x \cap \omega_2)) \\ &= \ln(B \cap (x \cap \omega_2 \times x \cap \omega_2)) \;=\; \ln(b_{x \cap \omega_2}) \;=\; H(x \cap \omega_2)\ . \end{aligned}$$

for each $x \in S$.

Now we have proved $(\star)$, and this completes the proof of the claim. $\dashv$

This completes the proof of Theorem 4.1. $\square$

We prove Theorem 1.2:

*Proof of Thm.1.2.* Assume that $\lambda \geq \omega_2$ and $I$ is a proper $\omega_2$-ideal over $\mathcal{P}_{\omega_2}\lambda$. Note that $I$ is precipitous by Fact 2.1. Let $G$ be a $\mathbb{P}_I$-generic filter over $V$, $M$ be the transitive collapse of the ultrapower of $V$ by $G$, and let $j : V \to M$ be the ultrapower map. By the elementarity of $j$, it suffices to show that $\mathsf{CB}$ does not hold in $M$. Let $\delta$ be ${\omega_2}^V$. Note that $\delta < j(\delta) = {\omega_2}^M$.

In $V$, define $H : (\mathcal{P}_{\omega_1}\delta)^V \to \omega_1$ as in the proof of Theorem 4.1. Let $H^* := j(H) \cap (\mathcal{P}_{\omega_1}\delta)^M \in M$. We show that if $\gamma \in [\delta, {\omega_2}^M)$ then the set $\{x \in (\mathcal{P}_{\omega_1}\gamma)^M \mid H^*(x \cap \delta) \geq \text{o.t.}\, x\}$ is stationary in $M$. By Lemma 4.2 this suffices.

Take an arbitrary $\gamma \in [\delta, {\omega_2}^M)$. First note that $H^* \restriction (\mathcal{P}_{\omega_1}\delta)^V = H$ because $j$ does not move elements of $(\mathcal{P}_{\omega_1}\delta)^V$. Hence, in $V[G]$, the set $\{x \in (\mathcal{P}_{\omega_1}\gamma)^{V[G]} \mid H^*(x \cap \delta) \geq \text{o.t.}\, x\}$ is stationary by the proof of Theorem 4.1. But $(\mathcal{P}_{\omega_1}\gamma)^M$ contains a club in $(\mathcal{P}_{\omega_1}\gamma)^{V[G]}$ because $|\gamma| = \omega_1$ in $M$. Hence, in $M$, the set $\{x \in (\mathcal{P}_{\omega_1}\delta)^M \mid H^*(x \cap \delta) \geq \text{o.t.}\, x\}$ is stationary.

This completes the proof. $\square$

# 5 Semiproper ideals and $\diamondsuit^*_{\omega_1}$

Theorem 1.3 follows the following fact and theorem:

**Fact 5.1** (Sakai [3])**.** *If there exists a semiproper $\omega_2$-ideal over $\mathcal{P}_{\omega_2}\lambda$ for some $\lambda \geq \omega_2$ then Chang's Conjecture holds.*

**Theorem 5.2** (Someone)**.** *Chang's Conjecture implies that $\diamondsuit^*_{\omega_1}$ does not hold.*

Here we give the proof of Theorem 5.2:

*Proof of Thm.5.2.* Assume that Chang's Conjecture holds and that there is a $\diamondsuit^*_{\omega_1}$-sequence $\langle \mathcal{F}_\alpha \mid \alpha \in \omega_1 \rangle$. Take a pairwise distinct sequence $\langle F_\gamma \mid \gamma \in \omega_2 \rangle$. For each $\gamma \in \omega_2$ let $C_\gamma \subseteq \omega_1$ be a club such that $C \subseteq \{\alpha \in \omega_1 \mid F_\gamma \restriction \alpha \in \mathcal{F}_\alpha\}$.

Define partial functions $P$ and $Q$ on $\omega_2 \times \omega_2$ as follows. For each distinct $\gamma, \delta \in \omega_2$ let $P(\gamma, \delta)$ be the least $\beta \in \omega_1$ such that $F_\gamma(\beta) \neq F_\delta(\beta)$. For each $\gamma \in \omega_2$ and each $\beta \in \omega_1$ let $Q(\gamma, \beta)$ be $\min(C_\gamma \setminus (\beta + 1))$.

By Chang's Conjecture there exists an elementary submodel $M$ of the structure $\langle \omega_2, P, Q \rangle$ such that $|M \cap \omega_1| = \omega$ and $|M| = \omega_1$. Let $\alpha := \sup(M \cap \omega_1) \in \omega_1$. Then for every $\gamma \in M$, $\alpha \in C_\gamma$ because $M$ is closed under $Q$ and $C_\gamma$ is a club. Hence $F_\gamma \restriction \alpha \in \mathcal{F}_\alpha$ for every $\gamma \in M$. Moreover $F_\gamma \restriction \alpha \neq F_\delta \restriction \alpha$ for every distinct $\gamma, \delta \in M$ because $M$ is closed under $P$. Hence $|\mathcal{F}_\alpha| \geq |M| = \omega_1$. This contradicts that $\mathcal{F}_\alpha$ is countable. $\square$

# References


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