Preservation of $\neg\mathsf{CG}$ by finite support product of Cohen forcing

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We show that the negation of club guessing principle is preserved by a finite support product of Cohen forcings. Thus the negation of club guessing principle gives no upper bound to 2^ω.

First we fix our notations:

Definition 1. *Let* CG*, the club guessing principle, be the following principle:*

$\mathsf{CG} \;\equiv\;$ *There exists a sequence* $\langle b_\alpha \mid \alpha \in \mathrm{Lim}(\omega_1)\rangle$ *with the following properties:*

(i) b_α *is an unbounded subset of* α *for each* $\alpha \in \mathrm{Lim}(\omega_1)$,

(ii) *for every club* $C \subseteq \omega_1$ *there exist* $\alpha \in \mathrm{Lim}(\omega_1)$ *and* $\gamma < \alpha$ *with* $b_\alpha \setminus \gamma \subseteq C$.

A sequence $\langle b_\alpha \mid \alpha \in \mathrm{Lim}(\omega_1)\rangle$ *satisfying* (i) *and* (ii) *above is called a* CG*-sequence.*

Definition 2. *For a set* X *let* $Q(X)$ *be the poset of all finite partial functions* $p : X \times \omega \to 2$ *ordered by reverse inclusions. (Let* $Q(\emptyset)$ *be the trivial poset* $\{\emptyset\}$*.)*

We prove the following:

Proposition 3. *Assume* $\neg\mathsf{CG}$*. Then* $\Vdash_{Q(X)}$ *"* $\neg\mathsf{CG}$ *" for every set* X*.*

To prove the above proposition we make some easy preparations. First we make a preparation on $Q(X)$. Note that $Q(X)$ has the c.c.c. The following is standard:

Lemma 4. *Let* X *be a set,* α *be a countable ordinal and* $\dot{b}$ *be a* $Q(X)$*-name of a subset of* α*. Then there exists a countable* $Y \subseteq X$ *such that for every* $\beta < \alpha$ *and every* $p \in Q(X)$ *the following hold:*

(1) *If* $p \Vdash_{Q(X)}$ *"*$\beta \in \dot{b}$*" then* $p \restriction Y \times \omega \Vdash_{Q(X)}$ *"*$\beta \in \dot{b}$*".*

(2) *If* $p \Vdash_{Q(X)}$ *"*$\beta \notin \dot{b}$*" then* $p \restriction Y \times \omega \Vdash_{Q(X)}$ *"*$\beta \notin \dot{b}$*".*

Proof. For each $\beta < \alpha$ take a maximal antichain $R_\beta \subseteq Q(X)$ which consists of conditions deciding whether $\beta \in \dot{b}$ or not. R_β is countable because $Q(X)$ has the c.c.c. Let R be $\bigcup_{n\in\omega} R_\beta$. Because α is countable R is countable, too.

Then we can take a countable $Y \subseteq X$ such that $R \subseteq Q(Y)$. We show that this Y witnesses the lemma. Take an arbitrary $\beta < \alpha$ and an arbitrary $p \in Q(X)$. We prove only (2). (1) can be proved in the same way. (In the proof of Prop.3 we only use (2).)

We show that if $p \restriction Y \times \omega \not\Vdash_{Q(X)}$ "$\beta \notin \dot{b}$" then $p \not\Vdash_{Q(X)}$ "$\beta \notin \dot{b}$". Suppose that $p \restriction Y \times \omega \not\Vdash_{Q(X)}$ "$\beta \notin \dot{b}$". Then there exists $r \in R_\beta$ which is compatible with $p \restriction Y \times \omega$ and forces that $\beta \in \dot{b}$. Note that p and r are compatible because $\mathrm{dom}(r) \subseteq Y \times \omega$, and $p \restriction Y \times \omega$ and r are compatible. Therefore $p \not\Vdash_{Q(X)}$ "$\beta \notin \dot{b}$". □

Next we give a preparation on club subsets of ω_1:

Notation 5. *Let $\mathcal{C}$ denote the set of all club subsets of ω_1. For a subset $\mathcal{C}'$ of $\mathcal{C}$ we say that $\mathcal{C}'$ is $\subseteq$-dense if for every $C \in \mathcal{C}$ there exists $C' \in \mathcal{C}'$ such that $C' \subseteq C$.*

Note that if $\mathcal{C}'$ is $\subseteq$-dense then the property (ii) in the definition of CG is equivalent to the following:

(ii)' For every $C' \in \mathcal{C}'$ there exist $\alpha \in \mathrm{Lim}(\omega_1)$ and $\gamma < \alpha$ with $b_\alpha \setminus \gamma \subseteq C'$.

The following is easy:

Lemma 6. *Suppose that $\langle \mathcal{C}_n \mid n \in \omega \rangle$ is a sequence of subsets of $\mathcal{C}$ with $\bigcup_{n\in\omega} \mathcal{C}_n = \mathcal{C}$. Then there exists $n \in \omega$ such that $\mathcal{C}_n$ is $\subseteq$-dense.*

Proof. For the contradiction, assume that $\mathcal{C}_n$ is not $\subseteq$-dense for every $n \in \omega$.

Then for each $n \in \omega$ we can take $C_n \in \mathcal{C}$ such that there are no $C \in \mathcal{C}_n$ with $C \subseteq C_n$. Let C^* be $\bigcap_{n\in\omega} C_n$.

Then $C^* \in \mathcal{C}$. On the other hand there are no $C \in \mathcal{C}$ such that $C \subseteq C^*$ by the construction of C^* and the fact that $\bigcup_{n\in\omega} \mathcal{C}_n = \mathcal{C}$. This is a contradiction. □

Now we proceed to the proof of the proposition:

Proof of Prop.3. We show that if there exists a set X with $\Vdash_{Q(X)}$ "CG" then CG holds in V. By the homogeneity of $Q(X)$ this suffices for the proposition. We work in V.

Suppose that there exists a set X with $\Vdash_{Q(X)}$ "CG". First of all note that ω_1 is absolute between V and $V^{Q(X)}$ because $Q(X)$ has the c.c.c. Let $\langle \dot{b}_\alpha \mid \alpha \in \mathrm{Lim}(\omega_1) \rangle$ be the sequence of $Q(X)$-names of a CG-sequence, that is, $\Vdash_{Q(X)}$ "$\langle \dot{b}_\alpha \mid \alpha \in \mathrm{Lim}(\omega_1) \rangle$ is a CG-sequence".

For each $\alpha \in \mathrm{Lim}(\omega_1)$ let Y_α be a countable subset of X obtained from Lemma 4 for $X, \alpha, \dot{b}_\alpha$. We claim the following. $\mathcal{C}$ below is the one defined in V, that is, the set of all club $C \subseteq \omega_1$ with $C \in V$.

Claim . *For every $C \in \mathcal{C}$ there exist $\alpha \in \mathrm{Lim}(\omega_1)$, $\gamma < \alpha$ and $p \in Q(Y_\alpha)$ such that $p \Vdash_{Q(X)}$ "$\dot{b}_\alpha \setminus \gamma \subseteq C$".*

Proof of Claim. Take an arbitrary $C \in \mathcal{C}$. Then there exist $\alpha \in \mathrm{Lim}(\omega_1)$, $\gamma < \alpha$ and $p \in Q(X)$ such that $p \Vdash_{Q(X)}$ "$\dot{b}_\alpha \setminus \gamma \subseteq C$". This is because $\langle \dot{b}_\alpha \mid \alpha \in \mathrm{Lim}(\omega_1)\rangle$ is a sequence of $Q(X)$-name of a CG-sequence, and C remains a club subset of ω_1 in $V^{Q(X)}$.

Then $p \Vdash_{Q(X)}$ "$\beta \notin \dot{b}_\alpha$" for every $\beta \geq \gamma$ with $\beta \notin C$. Thus, by the choice of Y_α, $p \restriction Y_\alpha \times \omega \Vdash_{Q(X)}$ "$\beta \notin \dot{b}_\alpha$" for every $\beta \geq \gamma$ with $\beta \notin C$. Therefore $p \restriction Y_\alpha \times \omega \Vdash_{Q(X)}$ "$\dot{b}_\alpha \setminus \gamma \subseteq C$".

Now α, γ and $p \restriction Y_\alpha \times \omega$ witness the claim. □(*Claim*)

By the claim above, for each $C \in \mathcal{C}$ take $\alpha_C \in \mathrm{Lim}(\omega_1)$, $\gamma_C < \alpha_C$ and $p_C \in Q(Y_{\alpha_C})$ such that

$$p_C \Vdash_{Q(X)} \text{"}\dot{b}_{\alpha_C} \setminus \gamma_C \subseteq C\text{"} .$$

Here note that $Q(Y_\alpha)$ is countable for each $\alpha \in \mathrm{Lim}(\omega_1)$ because Y_α is countable. For each $\alpha \in \mathrm{Lim}(\omega_1)$ fix an enumeration $\langle p^\alpha_n \mid n \in \omega\rangle$ of $Q(Y_\alpha)$. Moreover for each $C \in \mathcal{C}$ let n_C be such that $p_C = p^{\alpha_C}_{n_C}$.

Then, by Lemma 6, there exists $n^* \in \omega$ such that the set $\mathcal{C}^* = \{C \in \mathcal{C} \mid n_C = n^*\}$ is $\subseteq$-dense. Now for each $\alpha \in \mathrm{Lim}(\omega_1)$ define $b^*_\alpha \subseteq \alpha$ as follows:

$$b^*_\alpha \ = \ \{\beta < \alpha \mid (\exists p \in Q(X))\, p \leq p^\alpha_{n^*} \wedge p \Vdash_{Q(X)} \text{"}\beta \in \dot{b}_\alpha\text{"}\}$$

We show that $\langle b^*_\alpha \mid \alpha \in \mathrm{Lim}(\omega_1)\rangle$ is a CG-sequence.

First we check that b^*_α is an unbounded subset of α for each $\alpha \in \mathrm{Lim}(\omega_1)$. Clearly b^*_α is a subset of α because $\dot{b}_\alpha$ is a $Q(X)$-name of a subset of α. To show unboundedness, take an arbitrary $\delta < \alpha$. Then, because b^*_α is a $Q(X)$-name of an unbounded subset of α, there exist $p \leq p^\alpha_{n^*}$ and $\beta \geq \delta$ such that $p \Vdash_{Q(X)}$ "$\beta \in \dot{b}_\alpha$". Hence $\beta \in b^*_\alpha \setminus \delta \neq \emptyset$.

Next we check that for every $C \in \mathcal{C}$ there exist $\alpha \in \mathrm{Lim}(\omega_1)$ and $\gamma < \alpha$ with $b^*_\alpha \setminus \gamma \subseteq C$. Because $\mathcal{C}^*$ is $\subseteq$-dense it suffices to show that such α and γ exist for every $C \in \mathcal{C}^*$.

Take an arbitrary $C \in \mathcal{C}^*$. We show that $b^*_{\alpha_C} \setminus \gamma_C \subseteq C$. First note that $n^* = n_C$ and thus that $p^{\alpha_C}_{n^*} = p^{\alpha_C}_{n_C} = p_C$. So

$$p^{\alpha_C}_{n^*} \Vdash_{Q(X)} \text{"}\dot{b}_{\alpha_C} \setminus \gamma_C \subseteq C\text{"}$$

by the choice of α_C, γ_C and p_C. Hence if $p \leq p^{\alpha_C}_{n^*}$, $\beta \geq \gamma_C$ and $p \Vdash_{Q(X)}$ "$\beta \in \dot{b}_{\alpha_C}$" then $\beta \in C$. Therefore $b^*_{\alpha_C} \setminus \gamma_C \subseteq C$ by the construction of $b^*_{\alpha_C}$.

This completes the proof. □