COMPLETELY BOUNDING PRINCIPLE ON $\mathcal{P}_\kappa\lambda$

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1. Introduction

Definition 1.1. *Let κ be a regular uncountable cardinal, let λ be a cardinal with $\lambda \geq \kappa$ and let S be a stationary subset of $\mathcal{P}_\kappa\lambda$. Then $\mathrm{CB}_{\kappa\lambda}(S)$, the completely bounding principle on S, is the following principle:*

> *If $f : S \to \kappa$ is a function such that $f(x) < \mathrm{ot}(x)^+$ for every $x \in S$ then there are an $\alpha < \lambda^+$ and a surjection $\pi : \lambda \to \alpha$ such that the set $\{x \in S \mid \mathrm{ot}(\pi``x) \leq f(x)\}$ is nonstationary in $\mathcal{P}_\kappa\lambda$.*

Let $\mathrm{CB}_{\kappa\lambda}$ denote $\mathrm{CB}_{\kappa\lambda}(\mathcal{P}_\kappa\lambda)$.

We prove the following:

Theorem 1.2. *Let κ be a regular uncountable cardinal and λ be a cardinal with $\lambda \geq \kappa^{+\omega}$. Then $\mathrm{CB}_{\kappa\lambda}$ does not hold.*

Theorem 1.3. *Let λ be a cardinal with $\lambda > \omega_1$. Then $\mathrm{CB}_{\omega_1\lambda}$ does not hold.*

This note is constructed as follows:
§1: This section. We present notations and facts used in this note.
§2: We study basics on the completely bounding principle.
§3: We prove Theorem 1.2.
§4: We prove Theorem 1.3.

Notations and Facts

Let $\mathcal{M}$ be a structure which has definable well-ordering of its universe. For each $X \subseteq \mathcal{M}$, $\mathrm{Skull}^{\mathcal{M}}(X)$ denotes the Skolem hull of X in $\mathcal{M}$, that is, $\mathrm{Skull}^{\mathcal{M}}(X)$ is the smallest elementary submodel of $\mathcal{M}$ including X. We use the following lemma:

Lemma 1.4. *Let θ be a regular uncountable cardinal, Δ be a well-ordering of $\mathcal{H}_\theta$ and let $\mathcal{M}$ be the structure $\langle \mathcal{H}_\theta, \in, \Delta \rangle$. Moreover let M be an elementary submodel of $\mathcal{M}$, let γ be an ordinal in M and let X be a nonempty subset of γ. Let $N := \mathrm{Skull}^{\mathcal{M}}(M \cup X)$. Then, for every regular cardinal δ with $\gamma < \delta \in M$, $\sup(N \cap \delta) = \sup(M \cap \delta)$.*

Proof. First we prove the following claim:

Claim 1.4.1. *$N = \{f(s) \mid s \in {}^{<\omega}X \wedge f$ is a function on ${}^{<\omega}\gamma \wedge f \in M\}$.*

Proof of Claim. Let $N' := \{f(s) \mid s \in {}^{<\omega}X \wedge f$ is a function on ${}^{<\omega}\gamma \wedge f \in M\}$. Clearly $N' \subseteq N$. We show that $N \subseteq N'$. It is easy to see that $M \cup X \subseteq N'$. Hence it suffices to show that $N' \prec \mathcal{M}$.

We use Tarski-Vaught's criterion. Assume that φ is a formula, $a_1, \ldots, a_n \in N'$ and $\mathcal{M} \models \exists u \varphi(u, a_1, \ldots, a_n)$. We must show that there is an $a^* \in N'$ such that $\mathcal{M} \models \varphi(a^*, a_1, \ldots, a_n)$. Let $s_1, \ldots, s_n \in {}^{<\omega}X$ and $f_1, \ldots, f_n \in M$ be such that f_k

is a function on ${}^{<\omega}\gamma$ and $f_k(s_k) = a_k$ for each k. Let $s^* := s_1{}^\frown s_2{}^\frown \dots {}^\frown s_n$. For each k, take an function $g_k \in M$ on ${}^{<\omega}\gamma$ with $g_k(s^*) = s_k$ and let $h_k := f_k \circ g_k$. Then $h_k \in M$ and $h_k(s^*) = a_k$ for each k.

Now define a function h^* on ${}^{<\omega}\gamma$ as follows. For each $s \in {}^{<\omega}\gamma$, if $\mathcal{M} \models \exists u \varphi(u, h_1(s), \dots, h_n(s))$ then let $h^*(s)$ be the Δ-least a such that $\mathcal{M} \models \varphi(a, h_1(s), \dots, h_n(s))$. Otherwise let $h^*(s) := 0$. Then h^* belongs to $\mathcal{M}$ and is definable in $\mathcal{M}$ from parameters $h_1, \dots, h_n$. Hence $h^* \in M$ by the elementarity of M. Let $a^* := h^*(s^*)$. Then $a^* \in N'$. Moreover recall that $h_k(s^*) = a_k$. Hence $\mathcal{M} \models \varphi(a^*, a_1, \dots, a_n)$ by the construction of h^* and the assumption that $\mathcal{M} \models \exists u \varphi(u, a_1, \dots, a_n)$. □*Claim*

Let δ be a regular cardinal with $\gamma < \delta \in M$. Take an arbitrary $\alpha \in N \cap \gamma$. We show that $\alpha < \sup(M \cap \delta)$. By the claim above, there is a function $f \in M$ on ${}^{<\omega}\gamma$ and an $s \in {}^{<\omega}X$ such that $f(s) = \alpha$. Then $\alpha \le \sup(\mathrm{ran}(f) \cap \delta)$. But $\sup(\mathrm{ran}(f) \cap \delta) \in M \cap \delta$ by the elementarity of M and the regularity of δ. □

2. Basics

The following lemma is easy (but we prove):

Lemma 2.1. *Let κ be a regular uncoutable cardinal, λ be a cardinal with $\lambda \ge \kappa$ and let $\alpha < \lambda^+$. Moreover let π, π' be surjections from λ to α. Then the set $\{x \in \mathcal{P}_\kappa\lambda \mid \pi``x = \pi'``x\}$ contains a club in $\mathcal{P}_\kappa\lambda$. Hence the set $\{x \in \mathcal{P}_\kappa\lambda \mid \mathrm{ot}(\pi``x) = \mathrm{ot}(\pi'``x)\}$ contains a club.*

Proof. Let θ be a sufficiently large regular cardinal, Δ be a well-ordering of $\mathcal{H}_\theta$ and let $\mathcal{M}$ be the structure $\langle \mathcal{H}_\theta, \in, \Delta \rangle$. We show that if $x \in \mathcal{P}_\kappa\lambda$ and $\mathrm{Skull}^{\mathcal{M}}(x) \cap \lambda = x$ then $\pi``x = \pi'``x$. Clearly this suffices.

Assume that $x \in \mathcal{P}_\kappa\lambda$ and $\mathrm{Skull}^{\mathcal{M}}(x) \cap \lambda = x$. Let $M := \mathrm{Skull}^{\mathcal{M}}(x)$. Then $\pi``x \subseteq M \cap \alpha$ by the elementarity of $M$. On the other hand, for each $\beta \in M \cap \alpha$, there is an $\xi \in x$ with $\pi(\xi) = \beta$ because $\pi$ is surjective and $M$ is an elementary submodel of $\mathcal{M}$. Hence $M \cap \alpha = \pi``x$. Similarly $M \cap \alpha = \pi'``x$. Therefore $\pi``x = \pi'``x$. □

Proposition 2.2. *Let κ be a regular uncountable cardinal, let λ be a cardinal $\ge \kappa$ and let S be a stationary subset of $\mathcal{P}_\kappa\lambda$. If $\mathrm{NS}_{\kappa\lambda}|S$ is λ^+-saturated then $\mathrm{CB}_{\kappa\lambda}(S)$ holds.*

Definition 2.3. *Let κ be a regular uncountable cardinal, λ be a cardinal with $\lambda \ge \kappa$ and let S be a stationary subset of $\mathcal{P}_\kappa\lambda$. Then $\diamondsuit_{\kappa\lambda}(S)$ is the following principle:*

> *There is a sequence $\langle d_x \mid x \in S \rangle$ such that*
> (1) *$d_x : x \to 2$ for every $x \in S$,*
> (2) *for every function $F : \lambda \to 2$, the set $\{x \in S \mid F \restriction x = d_x\}$ is stationary in $\mathcal{P}_\kappa\lambda$.*

We call $\langle d_x \mid x \in S \rangle$ a $\diamondsuit_{\kappa\lambda}(S)$-sequence if it sutisfies (1) and (2) above.

Proposition 2.4. *Let κ be a regular uncountable cardinal, λ be a cardinal with $\lambda \ge \kappa$ and let S be a stationary subset of $\mathcal{P}_\kappa\lambda$. If $\diamondsuit_{\kappa\lambda}(S)$ holds then $\mathrm{CB}_{\kappa\lambda}(S)$ does not hold.*

3. Proof of Theorem 1.2

The proof of Theorem 1.2 splits into the following two lemmata:

Lemma 3.1. *Let κ be regular uncountable cardinal. Then $\mathrm{CB}_{\kappa\kappa^{+\omega}}$ does not hold.*

Lemma 3.2. *Let κ be regular uncountable cardinal and let $\lambda, \bar{\lambda}$ be cardinals with $\kappa \leq \lambda \leq \bar{\lambda}$. If $\mathrm{CB}_{\kappa\lambda}$ does not hold then $\mathrm{CB}_{\kappa\bar{\lambda}}$ does not hold.*

First we prove Lemma 3.1. We present two proofs. In the first proof we use the following fact:

Fact 3.3 (Shelah)**.** *Let κ be a regular uncountable cardinal and let λ be a cardinal with $\kappa < \lambda$ and $\mathrm{cf}(\lambda) = \omega$. Then $\mathcal{P}_\kappa\lambda$ splits into λ^ω disjoint stationary sets.*

The first proof of Lemma 3.1. Let $\lambda := \kappa^{+\omega}$. For each $\alpha < \lambda^+$, fix a surjection $\pi_\alpha : \lambda \to \alpha$. By Lemma 2.1, it suffices to find a function $f : \mathcal{P}_\kappa\lambda \to \kappa$ such that $f(x) < \mathrm{ot}(x)^+$ for every $x \in \mathcal{P}_\kappa\lambda$ and $\{x \in \mathcal{P}_\kappa\lambda \mid f(x) \geq \mathrm{ot}(\pi_\alpha``x)\}$ is stationary for every $\alpha < \lambda^+$.

By Fact 3.3, let $\langle S_\alpha \mid \alpha < \lambda^+\rangle$ be a partition of $\mathcal{P}_\kappa\lambda$ into pairwise disjoint stationary sets. Let f be a function on $\mathcal{P}_\kappa\lambda$ such that for each $x \in S_\alpha$, $f(x) = \mathrm{ot}(\pi_\alpha``x)$. Then for each $x \in \mathcal{P}_\kappa\lambda$, $f(x) < \mathrm{ot}(x)^+$. Moreover for each $\alpha < \lambda^+$, $\{x \in \mathcal{P}_\kappa\lambda \mid f(x) \geq \mathrm{ot}(\pi_\alpha``x)\}$ is staitionary because it includes S_α. This completes the proof. □

Next we present the second proof. We use Lemma 1.4 and a basic fact in PCF theory.

The second proof of Lemma 3.1. Let $\lambda := \kappa^{+\omega}$. For each $\alpha \in \lambda^+$, fix a surjection $\pi_\alpha : \lambda \to \alpha$. We construct a function $f : \mathcal{P}_\kappa\lambda \to \kappa$.

First take an increasing sequence of regular cardinals $\langle\lambda_n \mid n \in \omega\rangle$ such that

- $\kappa \leq \lambda_0$ and $\langle\lambda_n \mid n \in \omega\rangle$ is cofinal in λ,
- $\mathrm{tcf}(\Pi_{n\in\omega}\lambda_n/J_{\mathrm{bd}}) = \lambda^+$,

where J_{bd} is the bounded ideal over ω. Let $\langle\sigma_\alpha \mid \alpha < \lambda^+\rangle$ be a $<_{J_{\mathrm{bd}}}$-increasing cofinal sequence in $\Pi_{n\in\omega}\lambda_n/J_{\mathrm{bd}}$. Moreover, for each $x \in \mathcal{P}_\kappa\lambda$, let σ_x be the function on ω with $\sigma_x(n) = \sup(x \cap \lambda_n)$ and let α_x be the least $\alpha < \lambda^+$ such that $\sigma_x <_{J_{\mathrm{bd}}} \sigma_\alpha$. Now for each $x \in \mathcal{P}_\kappa\lambda$, let

$$f(x) := \sup\{\mathrm{ot}(\pi_\alpha``x) \mid \alpha \in \pi_{\alpha_x}``x\}\ .$$

Clearly $f(x) < \mathrm{ot}(x)^+$ for every $x \in \mathcal{P}_\kappa\lambda$. Hence it suffices to show that for every $\beta < \lambda^+$, $\{x \in \mathcal{P}_\kappa\lambda \mid f(x) \geq \mathrm{ot}(\pi_\beta``x)\}$ is stationary. Take an arbitrary $\beta < \lambda^+$ and an arbitrary function $p : {}^{<\omega}\lambda \to \lambda$. It suffices to find an $x^* \in \mathcal{P}_\kappa\lambda$ such that $x^* \cap \kappa \in \kappa$, $x^*$ is closed under $p$ and $f(x^*) \geq \mathrm{ot}(\pi_\beta``x)$.

Let θ be a sufficiently large regular cardinal, Δ be a well-ordering of $\mathcal{H}_\theta$ and let $\mathcal{M}$ be the structure $\langle\mathcal{H}_\theta, \in, \Delta\rangle$. Take an elementary submodel M of $\mathcal{M}$ with $\kappa, \lambda, p, \sigma_\beta \in M$ and $|M| < \kappa$. Let $\sigma^* := \sigma_{M\cap\lambda}$ and let $\alpha^* := \alpha_{M\cap\lambda}$. Note that $\alpha_* > \beta$ because $\sigma_\beta \in M$. Moreover note that $C := \{\xi < \kappa \mid \mathrm{Skull}^{\mathcal{M}}(M \cup \xi \cup \{{\pi_{\alpha^*}}^{-1}(\beta)\}) \cap \kappa = \xi\}$ contains a club in κ. Take an $\xi^* \in C$. Then let $N := \mathrm{Skull}^{\mathcal{M}}(\,M \cup \xi^* \cup \{{\pi_{\alpha^*}}^{-1}(\beta)\}\,)$ and let $x^* := N \cap \lambda$. Note that $x^* \cap \kappa = \xi^* \in \kappa$. Note also that x^* is closed under p because $p \in N \prec \mathcal{M}$. We show that $f(x^*) \geq \mathrm{ot}(\pi_\beta``x)$.

Let $n^* \in \omega$ be such that ${\pi_{\alpha^*}}^{-1}(\beta) \in \lambda_{n^*}$. Then $\xi^* \cup \{{\pi_{\alpha^*}}^{-1}(\beta)\} \subseteq \lambda_{n^*}$. Thus $\sup(N \cap \lambda_m) = \sup(M \cap \lambda_m)$ for every $m > n^*$ by Lemma 1.4. Hence $\sigma_{x^*} = \sigma^*$ and so $\alpha_{x^*} = \alpha^*$. Moreover $\beta \in \pi_{\alpha^*}``x^*$ because ${\pi_{\alpha^*}}^{-1}(\beta) \in N$. Hence $\beta \in \pi_{\alpha_{x^*}}``x^*$. Therefore $f(x^*) \geq \mathrm{ot}(\pi_\beta``x^*)$ by the construction of f. This completes the proof. □

Now we proceed to the proof of Lemma 3.2.

Proof of Lemma 3.2. Assume that $\mathrm{CB}_{\kappa\bar{\lambda}}$ holds. We show that $\mathrm{CB}_{\kappa\lambda}$ holds. Take an arbitrary function $f : \mathcal{P}_\kappa\lambda \to \kappa$ such that $f(x) < \mathrm{ot}(x)^+$ for every $x \in \mathcal{P}_\kappa\lambda$. We must find an $\alpha^* < \lambda^+$ and a surjection $\pi^* : \lambda \to \alpha^*$ such that $\{x \in \mathcal{P}_\kappa\lambda \mid f(x) < \mathrm{ot}(\pi^*``x)\}$ contains club in $\mathcal{P}_\kappa\lambda$.

Let $\bar{f} : \mathcal{P}_\kappa\bar{\lambda}$ be the function such that $\bar{f}(y) = f(y \cap \lambda)$ for each $y \in \mathcal{P}_\kappa\bar{\lambda}$. By $\mathrm{CB}_{\kappa\bar{\lambda}}$, take an $\bar{\beta} \in \bar{\lambda}^+$ and a surjection $\bar{\pi} : \bar{\lambda} \to \bar{\beta}$ such that $\bar{C} := \{y \in \mathcal{P}_\kappa\bar{\lambda} \mid \bar{f}(y) < \mathrm{ot}(\bar{\pi}``y)\}$ contains club. Then we can take an $W \subseteq \bar{\lambda}$ such that $\lambda \subseteq W$, $|W| = \lambda$ and $\bar{C} \cap \mathcal{P}_\kappa W$ contains a club in $\mathcal{P}_\kappa W$. Let $\rho$ be a bijection from $\lambda$ to $W$. Now let $\alpha^* := \mathrm{ot}(\bar{\pi}``W)$, let $\tau : \alpha^* \to \bar{\pi}``W$ be the increasing enumeration and let $\pi^* := \tau^{-1} \circ \bar{\pi} \circ \rho$. Then the following diagram commutes:

$$\begin{array}{ccc} \alpha^* & \xrightarrow{\;\tau\;} & \bar{\pi}``W \\ {\scriptstyle \pi^*}\uparrow & & \uparrow{\scriptstyle \bar{\pi}} \\ \lambda & \xrightarrow[\rho]{\quad} & W \end{array}$$

We show that α^* and π^* are ones desired. Note that π^* is a surjection because $\rho, \bar{\pi}$ and π^* are all surjective. Let $C := \{x \in \mathcal{P}_\kappa\lambda \mid \rho``x \in \bar{C} \cap \mathcal{P}_\kappa W \wedge (\rho``x) \cap \lambda = x\}$. Then it is easy to see that C contains a club in $\mathcal{P}_\kappa\lambda$. Moreover for each $x \in C$, the following holds:

$$f(x) \overset{(1)}{=} \bar{f}(\rho``x) \overset{(2)}{<} \mathrm{ot}(\bar{\pi}``(\rho``x)) \overset{(3)}{=} \mathrm{ot}(\pi^*``x)\ .$$

(1) holds by the construction of $\bar{f}$ and the fact that $(\rho``x) \cap \lambda = x$. (2) holds because $\rho``x \in \bar{C}$. For (3), first note that $\bar{\pi}``(\rho``x) = \tau``(\pi^*``x)$ because the above diagram commutes. Then (3) holds because τ is the increasing enumeration.

This completes the proof. □

4. Proof of Theorem 1.3

Theorem 1.3 is a corollary of Proposition 2.4 and the following fact by Shelah.

Fact 4.1 (Shelah)**.** *For every cardinal $\lambda > \omega_1$, $\diamondsuit_{\omega_1\lambda}$ holds.*

Here we present a direct proof of Theorem 1.3. We use the following lemma. For each regular uncountable cardinal δ, let $E^\delta_\omega := \{\eta \in \delta \mid \mathrm{cf}(\eta) = \omega\}$.

Lemma 4.2. *Let δ be a regular cardinal with $\delta > \omega_1$, let λ be a cardinal with $\lambda \geq \delta$ and let $\langle E_\xi \mid \xi < \omega_1\rangle$ be a partition of E^δ_ω into pairwise disjoint stationary subsets. For each countable set y, let ξ_y be such that $\sup(y \cap \delta) \in E_{\xi_y}$. (If $\sup(y \cap \delta)$ is a successor ordinal then let $\xi_y = 0$.)*

Then $T := \{y \in \mathcal{P}_{\omega_1}\lambda \mid \mathrm{ot}(y) \leq \xi_y\}$ is stationary in $\mathcal{P}_{\omega_1}\lambda$.

First we prove Theorem 1.3 using Lemma 4.2.

Proof of Theorem 1.3. Let $\langle E_\xi \mid \xi < \omega_1\rangle$ be a partition of $E^{\omega_2}_\omega$ into pairwise disjoint stationary subsets. For each countable set y, let ξ_y be such that $\sup(y \cap \omega_2) \in E_{\xi_y}$.

(If $\sup(y \cap \omega_2)$ is a successor ordinal then let $\xi_y = 0$.) We show that the function $x \mapsto \xi_x$ ($x \in \mathcal{P}_{\omega_1}\lambda$) witnesses $\neg\mathrm{CB}_{\omega_1\lambda}$.

Take an arbitrary $\alpha < \lambda^+$ and an arbitrary surjection $\pi : \lambda \to \alpha$. Let $T := \{y \in \mathcal{P}_{\omega_1}\lambda^+ \mid \xi_y \leq \mathrm{ot}(y) \wedge y \text{ is closed under } \pi\}$ and let $S := \{y \cap \lambda \mid y \in T\}$. Then, by Lemma 4.2, T is stationary in $\mathcal{P}_{\omega_1}\lambda^+$ and so S is stationary in $\mathcal{P}_{\omega_1}\lambda$. Hence it suffices to show that $\mathrm{ot}(\pi``x) \leq \xi_x$ for every $x \in S$. Assume $x \in S$. Take a $y \in T$ with $y \cap \lambda = x$. Then the following holds:

$$\mathrm{ot}(\pi``x) \overset{(1)}{\leq} \mathrm{ot}(y) \overset{(2)}{\leq} \xi_y \overset{(3)}{=} \xi_x \ .$$

(1) holds because $y \cap \lambda = x$ and y is closed under π. (2) holds because $y \in T$. (3) holds because $y \cap \lambda = x$ and $\lambda \geq \omega_2$.

This completes the proof. □

The rest of this section is devoted to the proof of Lemma 4.2. In the rest of this section, fix a regular cardinal $\delta > \omega_1$ and a cardinal $\lambda \geq \delta$. We need some preliminaries.

Notation 4.3. *Let $p : {}^{<\omega}\lambda \to \lambda$ be a function and let $z \subseteq \lambda$.*

$\mathrm{cl}_p(z)$ denotes the closure of z under p. By induction on $n \in \omega$, define $\mathrm{cl}_p^n(z)$ as follows:

- $\mathrm{cl}_p^0(z) = z$.
- $\mathrm{cl}_p^{n+1}(z) = \mathrm{cl}_p^n(z) \cup p``\,{}^{<n}(\mathrm{cl}_p^n(z))$.

All that we use are the following properties of cl_p^n. Assume that $z = \{\zeta_n \mid n \in \omega\}$ and let $z_n := \{\zeta_k \mid k \leq n\}$. Then

- $\mathrm{cl}_p(z) = \bigcup_{n\in\omega} \mathrm{cl}_p^n(z_n)$,
- $\mathrm{cl}_p^n(z_n) \subseteq \mathrm{cl}_p^{n+1}(z_{n+1})$,
- $\mathrm{cl}_p^n(z_n)$ is finite.

We use the following game:

Definition 4.4. *For each function $p : {}^{<\omega}\lambda \to \lambda$ and $\xi < \omega_1$, let $\Gamma_{p\,\xi}$ be the following two players game of length ω.*

In the n-th stage, first player I chooses an ordinal $\eta_n < \delta$. After that player II chooses an ordinal ζ_n with $\eta_n \leq \zeta_n < \delta$ and a strictly order preserving map $h_n : \mathrm{cl}_p^n(\{\zeta_k \mid k \leq n\}) \to \xi$ which extends h_{n-1}. If such ζ_n and h_n do not exist then the game is over.

I	η_0	η_1	$\cdots$	η_n	$\cdots$
II	ζ_0, h_0	ζ_1, h_1	$\cdots$	ζ_n, h_n	$\cdots$

If II could continue to play for ω-stages then II wins. Otherwise I wins.

Note that $\Gamma_{p\,\xi}$ is a closed game for II. Hence it is determined. Note also that if $\langle \eta_n, \zeta_n, h_n \mid n \in \omega\rangle$ is a play of $\Gamma_{p\,\xi}$ then $\bigcup_{n\in\omega} h_n$ is a strictly order preserving map from $\mathrm{cl}_p(\{\zeta_n \mid n \in \omega\})$. Hence $\mathrm{ot}(\mathrm{cl}_p(\{\zeta_n \mid n \in \omega\}) \leq \xi$.

Lemma 4.5. *For every function $p : {}^{<\omega}\lambda \to \lambda$, there is a $\xi < \omega_1$ such that II has a winning strategy for $\Gamma_{p\,\xi}$.*

Proof. Assume that $p : {}^{<\omega}\lambda \to \lambda$ is a function and I has a winning strategy σ_ξ for $\Gamma_{p\,\xi}$ for every $\xi < \omega_1$. We construct a play $\langle \eta_n, \zeta_n, h_n \mid n \in \omega\rangle$ of $\Gamma_{p\,\xi^*}$ in which I has played according to σ_{ξ^*}. If such a play was constructed then it contradicts that σ_{ξ^*} is a winning strategy.

Let θ be a sufficiently large regular cardinal, Δ be a well-ordering of $\mathcal{H}_\theta$ and let $\mathcal{M}$ be the structure $\langle \mathcal{H}_\theta, \in, \Delta \rangle$. Take a countable $M \prec \mathcal{M}$ with $\delta, \lambda, p, \langle \sigma_\xi \mid \xi < \omega_1 \rangle \in M$. Then let ξ^*, h^* and N be as follows:

- $\xi^* = \mathrm{ot}(M \cap \delta) < \omega_1$.
- $h^* : M \cap \lambda \to \xi^*$ is the transitive collapse.
- $N = \mathrm{Skull}^{\mathcal{M}}(M \cup \{\xi^*\})$.

Note that $\sigma_{\xi^*} \in N$. Note also that $\sup(N \cap \delta) = \sup(M \cap \delta)$ by Lemma 1.4.

By induction on n, we construct a play $\langle \eta_n, \zeta_n, h_n \mid n < \omega \rangle$ so that each ζ_n and h_n have the following additional properties:

- $\zeta_n \in M \cap \delta$.
- $h_n = h^* \restriction \mathrm{cl}_p^n(\{\zeta_k \mid k \leq n\})$.

Here note that if $\zeta_k \in M \cap \delta$ for each $k \leq n$ then $\mathrm{cl}_p^n(\{\zeta_k \mid k \leq n\}) \subseteq M \cap \lambda$ because M is closed under p. Moreover $h^* \restriction \mathrm{cl}_p^n(\{\zeta_k \mid k \leq n\}) \in N$ because $\mathrm{cl}_p^n(\{\zeta_k \mid k \leq n\})$ is a finite subset of N and $\mathrm{ran}(h^*) = \xi^* \subseteq N$.

Assume that $n \in \omega$ and $\langle \eta_m, \zeta_m, h_m \mid m < n \rangle$ has been constructed. First let $\eta_n := \sigma_{\xi^*}(\langle \zeta_m, h_m \mid m < n \rangle)$. Note that $\eta_n \in N \cap \delta$ because $\zeta_m, h_m \in N$ for each $m < n$ and $\sigma_{\xi^*} \in N$. Hence $\eta_n < \sup(N \cap \delta) = \sup(M \cap \delta)$. Let ζ_n be such that $\eta_n \leq \zeta_n \in M \cap \delta$ and let $h_n := h^* \restriction \mathrm{cl}_p^n(\{\zeta_k \mid k \leq n\})$. This completes the inductive construction. Moreover it is easy to check that $\langle \eta_n, \zeta_n, h_n \mid n \in \omega \rangle$ is a play in which I has played according to σ_{ξ^*}. □

Now we can prove Lemma 4.2.

Proof of Lemma 4.2. Take an arbitrary function $p : {}^{<\omega}\lambda \to \lambda$. We must find an element of T which is closed under p.

By Lemma 4.5, take a $\xi^* \in \omega_1$ such that II has a winning strategy σ^* for $\Gamma_{p\,\xi^*}$. Let θ be a sufficiently large regular cardinal, Δ be a well-ordering of $\mathcal{H}_\theta$ and let $\mathcal{M} := \langle \mathcal{H}_\theta, \in, \Delta \rangle$. Let M be an elementary submodel of $\mathcal{M}$ such that $\sigma^*, p, \delta, \lambda \in M$ and $M \cap \delta \in E_{\xi^*}$. Then take an increasing cofinal sequence $\langle \eta_n \mid n \in \omega \rangle$ in $M \cap \delta$ and let $\langle \zeta_n, h_n \rangle := \sigma^*(\langle \eta_m \mid m < n \rangle)$ for each $n \in \omega$. Finally let $y^* := \mathrm{cl}_p(\{\zeta_n \mid n \in \omega\})$. It suffices to show that $y^* \in T$.

Claim 4.5.1. $\sup(y^* \cap \delta) = M \cap \delta$.

Proof of Claim. First note that $\eta_n \leq \zeta_n \in M \cap \delta$ for each $n \in \omega$. $\zeta_n \in M \cap \delta$ follows from the fact that M is closed under σ^*. Hence $\{\zeta_n \mid n \in \omega\}$ is a cofinal subset of $M \cap \delta$. Thus $\sup(y^* \cap \delta) \geq M \cap \delta$. But $y^* \subseteq M$ because M is closed under p. Hence $\sup(y^* \cap \delta) \leq M \cap \delta$. Therefore $\sup(y^* \cap \delta) = M \cap \delta$. □*Claim*

Recall that $M \cap \delta \in E_{\xi^*}$. Hence $\xi_{y^*} = \xi^*$. On the other hand $\mathrm{ot}(y^*) \leq \xi^*$ because $\bigcup_{n \in \omega} h_n$ is a strictly order preserving map from y^* to ξ^*. Thus $\mathrm{ot}(y^*) \leq \xi_{y^*}$. Therefore y^* is an element of T which is closed under p. □