Martin's maximum and stationary reflection at ω_3

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Let $\mathsf{SR}([\omega_3]^\omega, \omega_3)$ be the following statement:

> For any stationary $X \subseteq [\omega_3]^\omega$ there exists $\delta < \omega_3$ such that $X \cap [\delta]^\omega$ is stationary in $[\delta]^\omega$.

We will prove that $\mathsf{SR}([\omega_3]^\omega, \omega_3)$ does not follow from Martin's Maximum (MM):

Theorem 1. *Assume that there is a supercompact cardinal. Then there exists a forcing extension in which* MM *holds, but* $\mathsf{SR}([\omega_3]^\omega, \omega_3)$ *fails.*

In fact we will prove that $\mathsf{SR}([\omega_3]^\omega, \omega_3)$ fails in the standard model of MM constructed in [1]. Our proof is somewhat similar as the main theorem of [3].

First we give a lemma, due to Shelah [2], which is used to construct a non-reflecting stationary subset of $[\omega_3]^\omega$. Below, for regular cardinals μ and ν with $\mu < \nu$ let E^ν_μ denote the set of all $\alpha < \nu$ with $\mathrm{cf}(\alpha) = \mu$.

Lemma 2 (Shelah [2])**.** *Let κ and λ be regular uncountable cardinals with $\kappa < \lambda$, and suppose that $\langle S_\alpha \mid \alpha < \kappa \rangle$ is a sequence of stationary subsets of E^λ_ω. Then the set*

$$X \;:=\; \{x \in [\lambda]^\omega \mid \sup(x) \in S_{\sup(x \cap \kappa)}\}$$

is stationary in $[\lambda]^\omega$.

We give a proof of this lemma for the completeness of this note. Fix regular cardinals κ and λ with $\kappa < \lambda$ until we finish the proof of the above lemma.

We will use a game. For each function $F : [\lambda]^{<\omega} \to \lambda$ and each $\alpha < \kappa$ let $\Game(F, \alpha)$ be the following two players game of length ω:

I	γ_0	γ_1	γ_2	$\cdots$
II	δ_0	δ_1	δ_2	$\cdots$

At the n-th inning, first Player I choose $\gamma_n < \lambda$, and then Player II choose $\delta_n < \lambda$ greater than γ_n. II wins if and only if

$$\mathrm{cl}_F(\{\delta_n \mid n < \omega\} \cup \alpha) \cap \kappa = \alpha\ ,$$

where $\mathrm{cl}_F(A)$ denotes the closure of A under F. We claim the following:

Lemma 3. *For any function $F : [\lambda]^{<\omega} \to \lambda$ there exists $\alpha \in E^\kappa_\omega$ such that II has a winning strategy for $\Game(F, \alpha)$.*

Proof. Take an arbitrary $F : [\lambda]^{<\omega} \to \lambda$. For the contradiction assume that I does not have a winning strategy for $\Game(F, \alpha)$ for any $\alpha \in E^\kappa_\omega$. Here note that each $\Game(F, \alpha)$ is an open-closed game. So each $\Game(F, \alpha)$ is determined. Thus I has a winning strategy τ_α for $\Game(F, \alpha)$ for each $\alpha \in E^\kappa_\omega$.

Let θ be a sufficiently large regular cardinal, and take $M \prec \langle \mathcal{H}_\theta, \in \rangle$ such that $\alpha^* := M \cap \kappa \in E^\kappa_\omega$ and such that $F, \langle \tau_\alpha \mid \alpha \in E^\kappa_\omega \rangle \in M$. By induction on $n < \omega$ let

$$\delta_n \; := \; \sup\{\tau_\alpha(\langle \delta_m \mid m < n \rangle) + 1 \mid \alpha \in E^\kappa_\omega\} \; \in \; M \; .$$

Then for each $n < \omega$ let

$$\gamma_n \; := \; \tau_{\alpha^*}(\langle \delta_m \mid m < n \rangle) \; < \; \delta_n \; .$$

Moreover let $A := \mathrm{cl}_F(\{\delta_n \mid n < \omega\} \cup \alpha^*)$.

Note that $\langle \gamma_n, \delta_n \mid n < \omega \rangle$ is a play of $\Game(F, \alpha^*)$ in which I has moved according to the winning strategy τ_{α^*}. Hence I wins with this play, that is, $A \cap \kappa \not\subseteq \alpha^*$. On the other hand, $A \subseteq M$ because $\{\delta_n \mid n < \omega\} \cup \alpha^* \subseteq M$, and $F \in M$. So $A \cap \kappa \subseteq M \cap \kappa = \alpha^*$. This is a contradiction. □

Proof of Lemma 2. Take an arbitrary function $F : [\lambda]^{<\omega} \to \omega$. We will find $x \in X$ closed under F.

By Lemma 3 we can take $\alpha \in E^\kappa_\omega$ such that II has a winning strategy τ for $\Game(F, \alpha)$. Then we can take $\gamma \in S_\alpha \setminus \kappa$ closed under τ and F because S_α is stationary. Take cofinal sequences $\langle \alpha_n \mid n < \omega \rangle$ and $\langle \gamma_n \mid n < \omega \rangle$ in α and γ, respectively. Moreover let $\delta_n := \tau(\langle \gamma_m \mid m \leq n \rangle)$ for each $n < \omega$, and let $x := \mathrm{cl}_F(\{\delta_n \mid n < \omega\} \cup \{\alpha_n \mid n < \omega\})$.

It suffices to show that $x \in X$. For this first note that $\gamma_n < \delta_n < \gamma$ for each $n < \omega$, where the latter inequality follows from the closure of γ under τ. Moreover γ is closed under F. So it follows that $\sup(x) = \gamma \in S_\alpha$. Here note also that $\sup(x \cap \kappa) = \alpha$ because τ is a winning strategy of II for $\Game(F, \alpha)$. Therefore $\sup(x \cap \kappa) \in E^\kappa_\omega$, and $\sup(x) \in S_{\sup(x \cap \kappa)}$. □

Now we prove Theorem 1. As we mentioned before, we will prove that $\mathsf{SR}([\omega_3]^\omega, \omega_3)$ fails in the standard model of MM constructed in [1]:

Proof of Theorem 1. Suppose that κ is a supercompact cardinal in V. Take a Laver function $F : \kappa \to V_\kappa$, and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ be the revised countable support iteration of semi-proper posets along F. For each $\alpha \leq \kappa$ let

G_α be a $\mathbb{P}_\alpha$-generic filter over V. Then MM holds in $V[G_\kappa]$. We will show that $\mathsf{SR}([\omega_3]^\omega, \omega_3)$ fails in $V[G_\kappa]$. Note that $\kappa = \omega_2$ in $V[G_\kappa]$. We work in $V[G_\kappa]$.

First note that there are unboundedly many $\beta < \kappa$ such that in $V[G_\beta]$, $\beta = \omega_2$, and $(\dot{\mathbb{Q}}_\beta)_{G_\beta}$ is the Nambda forcing. Thus there are unboundedly many $\beta < \kappa$ such that $\mathrm{cf}(\beta) = \omega$ and such that β is regular in V. Note also that $(E^{\kappa^+}_\beta)^V$ is a stationary subset of $E^{\kappa^+}_\omega$ for each such β because $\mathbb{P}_\kappa$ has the κ-c.c. Hence for each $\alpha < \kappa$ the set

$$S_\alpha \ := \ \{\gamma \in E^{\kappa^+}_\omega \mid \mathrm{cf}^V(\gamma) > \alpha\}$$

is stationary. Then by Lemma 2 the set

$$X \ := \ \{x \in [\kappa^+]^\omega \mid \sup(x) \in S_{\sup(x \cap \kappa)}\}$$

is stationary in $[\kappa^+]^\omega$. So it suffices to show that $X \cap [\delta]^\omega$ is non-stationary for any $\delta < \kappa^+$. Note that

$$\mathrm{cf}^V(\sup(x)) > \sup(x \cap \kappa)$$

for each $x \in X$.

Take an arbitrary $\delta < \kappa^+$. First suppose that $\mathrm{cf}(\delta) = \omega$. Then the set

$$Y \ := \ \{x \in [\delta]^\omega \mid \sup(x) = \delta \ \wedge \ \mathrm{cf}^V(\delta) \leq \sup(x \cap \kappa)\}$$

is club in $[\delta]^\omega$. But $\mathrm{cf}^V(\sup(x)) \leq \sup(x \cap \kappa)$ for each $x \in Y$. So $X \cap Y = \emptyset$. Thus $X \cap [\delta]^\omega$ is non-stationary.

Next suppose that $\mathrm{cf}(\delta) > \omega$. In V take a club $c \subseteq \delta$ with $\mathrm{o.t.}(c) \leq \kappa$. Moreover define a function $f : \delta \to \kappa$ by $f(\gamma) := \mathrm{o.t.}(c \cap \gamma)$. Then the set

$$Z \ := \ \{x \in [\delta]^\omega \mid \sup(x) \in \mathrm{Lim}(c) \ \wedge \ x \text{ is closed under } f\}$$

is club in $[\delta]^\omega$. Note that if $x \in Z$, then

$$\mathrm{cf}^V(\sup(x)) \leq \mathrm{o.t.}(c \cap \sup(x)) \leq \sup(x \cap \kappa) \ .$$

So $X \cap Z = \emptyset$, that is, $X \cap [\delta]^\omega$ is non-stationary. □

References

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[3] H. Sakai, *Semistationary and stationary reflection*, J. Symbolic Logic **73** (2008), no.1, 181–192.