Club Guessing on $\mathcal{P}_\kappa\lambda$

Hiroshi Sakai

Throughout this note, let κ and λ be regular cardinals with $\kappa \le \lambda$. We consider the following club guessing principle on $\mathcal{P}_\kappa\lambda$.

Definition 1. *For a stationary $S \subseteq \mathcal{P}_\kappa\lambda$, let $\mathrm{CG}(S)$ be the following principle:*

There is a sequence $\langle b_x \mid x \in S\rangle$ such that

(1) *b_x is an unbounded subset of x for each $x \in S$,*

(2) *for every club $B \subseteq \lambda$, $\{x \in S \mid b_x \subseteq B\}$ is stationary in $\mathcal{P}_\kappa\lambda$.*

We call $\langle b_x \mid x \in S\rangle$ satisfying (1) and (2) a $\mathrm{CG}(S)$-sequence.

For a regular cardinal $\delta < \kappa$, let

$$E^{\kappa\lambda}_\delta \;:=\; \{x \in \mathcal{P}_\kappa\lambda \mid \mathrm{cf}(\sup x) = \delta \;\wedge\; \sup x \notin x\}\ .$$

Note that $E^{\kappa\lambda}_\delta$ is stationary. We prove the following:

Proposition 2. *Suppose that δ is a regular cardinal with $\delta^+ < \kappa$. Then $\mathrm{CG}(S)$ holds for every stationary $S \subseteq E^{\kappa\lambda}_\delta$.*

This is a slight generalization of the well-known fact, discovered by Shelah, that if $\delta^+ < \kappa$ then every stationary subset of E^κ_δ carries a club guessing sequence. ($E^\kappa_\delta = \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \delta\}$.)

To prove Proposition 2, first we modify CG slightly:

Definition 3. *For a stationary $S \subseteq \mathcal{P}_\kappa\lambda$, let $\mathrm{CG}'(S)$ be the following principle:*

There is a sequence $\langle b_x \mid x \in S\rangle$ such that

(1′) *b_x is an unbounded subset of $\sup x$ for each $x \in S$,*

(2) *for every club $B \subseteq \lambda$, $\{x \in S \mid b_x \subseteq B\}$ is stationary in $\mathcal{P}_\kappa\lambda$.*

We call $\langle b_x \mid x \in S\rangle$ satisfying (1′) and (2) a $\mathrm{CG}'(S)$-sequence.

Lemma 4. *$\mathrm{CG}(S)$ is equivalent with $\mathrm{CG}'(S)$ for every stationary $S \subseteq \mathcal{P}_\kappa\lambda$.*

Prior to proving Lemma 4, we give a notation:

Notation . *For a club $B \subseteq \lambda$ and an ordinal $\alpha < \lambda$, let*

$$\mathrm{m}(B,\alpha) \;:=\; \begin{cases} \max(B \cap (\alpha+1)) & \cdots \text{ if } B \cap (\alpha+1) \neq \emptyset \\ 0 & \cdots \text{ otherwise} \end{cases}\ .$$

Proof of Lemma 4. Let S be a stationary subset of $\mathcal{P}_\kappa\lambda$. Clearly $\mathrm{CG}(S)$ implies $\mathrm{CG}'(S)$. So we prove the other direction.

Assume that $\mathrm{CG}'(S)$ holds. Let $\langle b_x \mid x \in S\rangle$ be a $\mathrm{CG}'(S)$-sequence. Then for each $x \in S$, let

$$a_x \;:=\; \{\min(x \setminus \beta) \mid \beta \in b_x\}\ .$$

We show that $\langle a_x \mid x \in S\rangle$ is a $\mathrm{CG}(S)$-sequence. Clearly $\langle a_x \mid x \in S\rangle$ satisfies (1) in Definition 1.

To show that $\langle a_x \mid x \in S\rangle$ satisfies (2), take an arbitrary club $B \subseteq \lambda$. Let

$$T \;:=\; \{x \in S \mid b_x \subseteq B \;\wedge\; \forall\alpha \in x,\ \mathrm{m}(B,\alpha) \in x\}\ .$$

Then T is stationary because $\langle b_x \mid x \in S\rangle$ is a $\mathrm{CG}'(S)$-sequence. We show that $a_x \subseteq B$ for every $x \in T$. On the contrary, assume that $x \in T$ and $a_x \not\subseteq B$. Let $\beta \in b_x$ be such that $\alpha := \min(x\setminus\beta) \notin B$. First note that $\beta \in B$ because $b_x \subseteq B$. Hence $\beta < \alpha$ and $\mathrm{m}(B,\alpha) \in [\beta,\alpha)$. Then, because $\mathrm{m}(B,\alpha) \in x$, $x \cap [\beta,\alpha) \neq \emptyset$. But this contradicts that $\alpha = \min(x \setminus \beta)$. Therefore $a_x \subseteq B$ for every $x \in T$.

This completes the proof. $\square$(*Lem. 4*)

Now we prove Proposition 2. By Lemma 4, it suffices to show that $\mathrm{CG}'(S)$ holds for every stationary $S \subseteq E^{\kappa\lambda}_\delta$. For this, all we have to do is to translate Shelah's proof of the above mentioned fact into our context.

Proof of Prop. 2. Let δ be a regular cardinal with $\delta^+ < \kappa$ and take an arbitrary stationary $S \subseteq E^{\kappa\lambda}_\delta$. By Lemma 4, it suffices to show that $\mathrm{CG}'(S)$ holds. For the contradiction, assume that $\mathrm{CG}'(S)$ does not hold.

By induction on $\xi < \delta^+$, take a club $B_\xi \subseteq \lambda$, an $S_\xi \subseteq S$ and a sequence $\langle b_{x,\xi} \mid x \in S_\xi\rangle$ as follows. We will take them so that

(i) $S \setminus S_\xi$ is nonstationary in $\mathcal{P}_\kappa\lambda$,

(ii) for each $x \in S_\xi$, $b_{x,\xi}$ is an unbounded subset of $\sup x$ with $0 \notin b_{x,\xi}$.

(Base step) Let $B_0 := \lambda$ and let $S_0 := S$. For each $x \in S$, let $b_{x,0}$ be an unbounded subset of $\sup x$ of order type δ with $0 \notin b_{x,0}$.

(Suc. step) Assume that B_ξ, S_ξ and $\langle b_{x,\xi} \mid x \in S_\xi\rangle$ have been defined to satisfy (i) and (ii) above. Then note that $\langle b_{x,\xi} \mid x \in S_\xi\rangle$ is not a $\mathrm{CG}'(S_\xi)$-sequence because $\mathrm{CG}'(S)$ does not hold. Let $B_{\xi+1}$ be a club of λ such that $B_{\xi+1} \subseteq B_\xi$ and $\{x \in S_\xi \mid b_{x,\xi} \subseteq B_{\xi+1}\}$ is nonstationary. Then let

$$S_{\xi+1} \;:=\; \{x \in S_\xi \mid b_{x,\xi} \not\subseteq B_{\xi+1} \;\wedge\; \sup x \in \mathrm{Lim}(B_{\xi+1})\}$$

and let

$$b_{x,\xi+1} \;:=\; \{\mathrm{m}(B_{\xi+1},\beta) \mid \beta \in b_{x,0} \;\wedge\; \mathrm{m}(B_{\xi+1},\beta) > 0\}$$

for each $x \in S_{\xi+1}$. Clearly $S_{\xi+1}$ and $\langle b_{x,\xi+1} \mid x \in S_{\xi+1}\rangle$ satisfies (i) and (ii).

(Limit step) Assume that ξ is a limit ordinal and that B_η, S_η and $\langle b_{x,\eta} \mid x \in S_\eta\rangle$ have been defined to satisfy (i) and (ii) for each $\eta < \xi$. Then let

$$\begin{aligned} B_\xi &:= \textstyle\bigcap_{\eta<\xi} B_\eta\ , \\ S_\xi &:= \{x \in \textstyle\bigcap_{\eta<\xi} S_\eta \mid \sup x \in \mathrm{Lim}(B_\xi)\}\ . \end{aligned}$$

Note that $S\setminus\bigcap_{\eta<\xi} S_\eta$ is nonstationary because $\xi<\kappa$ and $S\setminus S_\eta$ is nonstationary for each $\eta<\xi$. Hence $S\setminus S_\xi$ is nonstationary. Finally let

$$b_{x,\xi} \ := \ \{\mathrm{m}(B_\xi,\beta) \mid \beta\in b_{x,0} \ \wedge \ \mathrm{m}(B_\xi,\beta)>0\}$$

for each $x\in S_\xi$. Clearly $b_{x,\xi}$ is unbounded in $\sup x$.

Now note that $\bigcap_{\xi<\delta^+} S_\xi$ is stationary in $\mathcal{P}_\kappa\lambda$ because $\delta^+<\kappa$ and $S\setminus S_\xi$ is nonstationary for each $\xi<\delta^+$. Take an $x\in\bigcap_{\xi<\delta^+} S_\xi$. For each $\gamma<\delta$, let β_0^γ be the γ-th element of $b_{x,0}$ and let $\beta_\xi^\gamma := \mathrm{m}(B_\xi,\beta_0^\gamma)$ for each $\xi<\delta^+$. Then for each $\gamma<\delta$, $\langle\beta_\xi^\gamma \mid \xi<\delta^+\rangle$ is decreasing (in the wider sense) because $\langle B_\xi \mid \xi<\delta^+\rangle$ is $\subseteq$-decreasing. Moreover if $\xi<\delta^+$ then $b_{x,\xi}=\{\beta_\xi^\gamma \mid \gamma<\delta\}\setminus\{0\}$. Hence there is a $\gamma<\delta$ such that $0<\beta_\xi^\gamma\notin B_{\xi+1}$ because $x\in S_{\xi+1}$. Then $\beta_{\xi+1}^\gamma<\beta_\xi^\gamma$ for such γ. Thus there is a $\gamma<\delta$ with $\beta_{\xi+1}^\gamma<\beta_\xi^\gamma$ for each $\xi<\delta^+$. Therefore there is a $\gamma<\delta$ such that $\langle\beta_\xi^\gamma \mid \xi<\delta^+\rangle$ strictly decreases for δ^+-many times. This is a contradiction. □(*Prop. 2*)