Partial Stationary Reflection Principles

Toshimichi Usuba (薄葉 季路)
Nagoya University

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Stationary reflection

κ: infinite cardinal $> \omega_1$.

Definition 1.
WRP($[\kappa]^\omega$) $\equiv$ Every stationary set $S \subseteq [\kappa]^\omega$ reflects to some $X \subseteq \kappa$ with $|X| = \omega_1 \subseteq X$, that is, $S \cap [X]^\omega$ is stationary in $[X]^\omega$.

Fact 2 (Shelah, Velickovic)**.** *The following are equiconsistent:*

① *There is a weakly compact cardinal.*

② WRP($[\omega_2]^\omega$)*.*

Partial stationary reflection

Definition 3. Let $S \subseteq [\kappa]^\omega$ be stationary.

WRP(S) $\equiv$ Every stationary subset T of S reflects to some $X \subseteq \kappa$ with $|X| = \omega_1 \subseteq X$.

Fact 4 (Sakai)**.** *The statement "WRP(S) holds for some stationary $S \subseteq [\omega_2]^\omega$" is equiconsistent with ZFC.*

Theorem 5. *Suppose CH. Fix a stationary* $\mathcal{X} \subseteq [\kappa]^{\omega_1}$. *Then there is a poset* $\mathbb{P}$ *such that:*

① $\mathbb{P}$ *is* σ*-closed and has the* ω_2*-c.c. (so preserves the stationarity of* $\mathcal{X}$*).*

② *In* $V^{\mathbb{P}}$*, there is a stationary* $S \subseteq [\kappa]^{\omega}$ *such that every stationary subset* $T \subseteq S$ *reflects to some* $X \in \mathcal{X}$*. Hence* WRP(S) *holds.*

$\Rightarrow$ WRP(S) for some $S \subseteq [\kappa]^{\omega}$ is not a large cardinal property even if $\kappa > \omega_2$.

Definition 6. κ: regular

$\mathrm{RP}([\kappa]^\omega) \equiv$ Every stationary $S \subseteq [\kappa]^\omega$ reflects to some $X \subseteq \kappa$ with $|X| = \omega_1 \subseteq X$ and $\mathrm{cf}(\sup(X)) = \omega_1$.

Fact 7 (Krueger). *Relative to a certain large cardinal assumption, it is consistent that* $\mathrm{WRP}([\omega_2]^\omega)$ *but* $\neg\mathrm{RP}([\omega_2]^\omega)$.

It is unknown the consistency of $\mathrm{WRP}([\kappa]^\omega) \wedge \neg\mathrm{RP}([\kappa]^\omega)$ for $\kappa > \omega_2$.

But $\mathrm{WRP}(S) \wedge \neg\mathrm{RP}(S)$ for some $S \subseteq [\kappa]^\omega$ is consistent.

Proof of Proposition

First, define the poset $\mathbb{P}$ as follows: $\mathbb{P}$ is the set of all countable set p of $[\kappa]^\omega$ such that $\bigcup(p) \in p$. $p \leq q \iff p \supseteq q$ and for every $x \in p$, $x \subseteq \bigcup q \Rightarrow x \in q$.

It is easy to check that $\mathbb{P}$ is σ-closed and satisfies the ω_2-c.c. If G is $(V, \mathbb{P})$-generic, then $S = \bigcup G$ is stationary in $[\kappa]^\omega$.

By the genericity of S, we can construct an iteration of club shootings $\mathbb{Q}$ which is σ-Baire, has the ω_2-c.c., and destroys all stationary subsets of S which do not reflect to every $X \in \mathcal{X}$. Moreover, in V, one can find a σ-closed dense subset of $\mathbb{P} * \mathbb{Q}$.

Remark 8 (Shelah, Todorcevic)**.**
κ: *regular* $\wedge$ $\mathrm{WRP}([\kappa]^\omega) \Rightarrow \kappa^\omega = \kappa$.

Proposition 9. *Suppose* $\mathrm{WRP}(S)$ *for some* $S \subseteq [\kappa]^\omega$. *Then every c.c.c. poset preserves* $\mathrm{WRP}(S)$. *In particular, "*$\mathrm{WRP}(S) \wedge 2^\omega$ *is arbitrary large" is consistent.*

Proof

Pick $p \in \mathbb{P}$ and $p \Vdash$ "$\dot{T} \subseteq S$ is stationary". Let

$$T = \{x \in S : \exists q \leq p \, (q \Vdash x \in \dot{T})\}.$$

T is stationary, so reflects to some $X \in [\kappa]^{\omega_1}$.

Fix a bijection $\pi : \omega_1 \to X$ and let

$$E = \{\alpha < \omega_1 : \pi``\alpha \in T \cap [X]^{\omega}\}.$$

E is stationary in ω_1. Then, since $\mathbb{P}$ has the c.c.c., one can find $r \leq p$ such that $r \Vdash$ "$\{\alpha \in E : \pi``\alpha \in \dot{T}\}$ is stationary", hence $q \Vdash$ "$\dot{T} \cap [X]^{\omega}$ is stationary".

Simultaneous stationary reflection

Definition 10.
$\mathrm{WRP}^2([\kappa]^\omega) \equiv$ Every stationary sets $S_0, S_1 \subseteq [\kappa]^\omega$ reflect to some $X \subseteq \kappa$ with $|X| = \omega_1 \subseteq X$ simultaneously, that is, both $S_0 \cap [X]^\omega$ and $S_1 \cap [X]^\omega$ are stationary in $[X]^\omega$.

Remark 11. *κ: weakly compact.*
Then $\mathrm{WRP}^2([\omega_2]^\omega)$ *holds in* $V^{\mathrm{col}(\omega_1, <\kappa)}$. *Hence* $\mathrm{WRP}^2([\omega_2]^\omega)$ *is still equiconsistent with the existence of a weakly compact cardinal.*

Simultaneous partial stationary reflection

Definition 12. Let $S_0, S_1 \subseteq [\kappa]^\omega$ be stationary. WRP$(S_0, S_1) \equiv$ Every stationary subsets $T_0 \subseteq S_0, T_1 \subseteq S_1$ reflect to some $X \subseteq \kappa$ with $|X| = \omega_1 \subseteq X$ simultaneously.

So WRP$([\kappa]^\omega, [\kappa]^\omega) \equiv$ WRP$^2([\kappa]^\omega)$.

Definition 13. κ: regular.

$\square(\kappa) \equiv$ there is $\langle C_\alpha : \alpha < \kappa \rangle$ such that:

① $C_\alpha \subseteq \alpha$ is a club in α.

② For every $\beta \in \lim(C_\alpha)$, $C_\beta = C_\alpha \cap \beta$.

③ There is no club C in κ such that $C \cap \alpha = C_\alpha$ for $\alpha \in \lim(C)$.

Fact 14 (Jensen). *There following are equiconsistent:*

① *There is a weakly compact cardinal.*

② $\square(\omega_2)$ *fails.*

Proposition 15. λ: *regular with* $\omega_2 \leq \lambda \leq \kappa$.
If $\mathrm{WRP}(S_0, S_1)$ *holds for some stationary* $S_0, S_1 \subseteq [\kappa]^\omega$, *then* $\square(\lambda)$ *fails.*

Corollary 16. *There following are equiconsistent:*

① *There is a weakly compact cardinal.*

② $\mathrm{WRP}([\omega_2]^\omega)$ *holds.*

③ $\mathrm{WRP}^2([\omega_2]^\omega)$ *holds.*

④ $\mathrm{WRP}(S_0, S_1)$ *holds for some stationary* $S_0, S_1 \subseteq [\omega_2]^\omega$.

Lemma 17. λ: *regular with* $\omega_2 \leq \lambda \leq \kappa$.
$S_0, S_1 \subseteq [\kappa]^\omega$: *stationary.*
Then there are stationary $T_0 \subseteq S_0$ *and* $T_1 \subseteq S_1$ *such that if* T_0 *and* T_1 *reflect to* $X \in [\kappa]^{\omega_1}$, *then* $\mathrm{cf}(\sup(X \cap \lambda)) = \omega_1$.

Proof of Lemma in the case $\kappa = \lambda$

Let S_0, S_1 be stationary and suppose to the contrary that for every stationary $T_0 \subseteq S_0$ and $T_1 \subseteq S_1$, there is $X \subseteq \kappa$ such that

① $|X| = \omega_1 \subseteq X$.

② $\sup(X \cap \lambda) \notin X$ and $\mathrm{cf}(\sup(X \cap \lambda)) = \omega$.

③ both $T_0 \cap [X]^\omega$ and $T_1 \cap [X]^\omega$ are stationary in $[X]^\omega$.

For each $\alpha < \kappa$ with $\mathrm{cf}(\alpha) = \omega$, fix $\langle \gamma_i^\alpha : i < \omega \rangle$ an increasing sequence with limit α.

For $n < 2$, $i < \omega$, and $\delta < \kappa$, let

$$S_{n,i,\delta} = \{x \in S_n : \delta = \min(x \setminus \gamma_i^{\sup(x)})\}.$$

Then for every $n < 2$ and $i < \omega$ there is $\delta < \kappa$ such that $S_{n,i,\delta}$ is stationary.

Claim 18. *For every $i < \omega$ and $\delta_0, \delta_1 < \kappa$, if S_{0,i,δ_0} and S_{1,i,δ_1} are stationary then $\delta_0 = \delta_1$.*

This means that if $S_{0,i,\delta}$ and $S_{0,i,\delta'}$ are stationary, then $\delta = \delta'$. This is impossible.

Proof of Proposition in the case $\lambda = \kappa$

If $\mathrm{WRP}(S_0, S_1)$ holds for some stationary $S_0, S_1 \subseteq [\kappa]^\omega$, then $\square(\kappa)$ fails.

Let $\langle c_\alpha : \alpha < \kappa \rangle$ be a coherent sequence.

For $\alpha < \kappa$ and $n < 2$, let

$$S_{n,\alpha} = \{x \in S_n : C_{\sup(x)} \cap \sup(x \cap \alpha) = C_\alpha \cap \sup(x \cap \alpha)\}.$$

For $n < 2$,

$$A_n = \{\alpha < \kappa : S_{n,\alpha} \text{ is stationary}\}.$$

Claim 19. *A_n is unbounded in κ.*

Claim 20. *For each* $\alpha \in A_0$ *and* $\beta \in A_1$*, if* $\alpha \leq \beta$ *then* $C_\alpha = C_\beta \cap \alpha$*, and* $\beta \leq \alpha$ *then* $C_\beta = C_\alpha \cap \beta$*.*

By WRP(S_0, S_1), there is $X \subseteq \kappa$ such that cf(sup$(X)) = \omega_1$, $\alpha, \beta \in X$, and both $S_{0,\alpha} \cap [X]^\omega$ and $S_{1,\beta} \cap [X]^\omega$ are stationary. Then for almost all $x \in S_{0,\alpha} \cap [X]^\omega$,

$C_\alpha \cap \mathrm{sup}(x \cap \alpha) = C_{\mathrm{sup}(x)} \cap \mathrm{sup}(x \cap \alpha) = C_{\mathrm{sup}(X)} \cap \mathrm{sup}(x \cap \alpha)$.

Since $\{\mathrm{sup}(x \cap \alpha) : x \in S_{0,\alpha} \cap [X]^\omega\}$ is unbounded in sup$(X \cap \alpha)$,

$$C_\alpha \cap \mathrm{sup}(X \cap \alpha) = C_{\mathrm{sup}(X)} \cap \mathrm{sup}(X \cap \alpha).$$

Similarly,

$$C_\beta \cap \mathsf{sup}(X \cap \beta) = C_{\mathsf{sup}(X)} \cap \mathsf{sup}(X \cap \beta).$$

So

$$C_\beta \cap \mathsf{sup}(X \cap \alpha) = C_\alpha \cap \mathsf{sup}(X \cap \alpha).$$

Since the set of $X \in [\kappa]^{\omega_1}$ at which $S_{0,\alpha}$ and $S_{1,\beta}$ reflect is stationary, we have $C_\alpha = C_\beta \cap \alpha$.

Finally, let $C = \bigcup\{C_\alpha : \alpha \in A_0\}$. Then $C \cap \alpha = C_\alpha$ for every $\alpha \in \mathsf{lim}(C)$. Hence $\langle C_\alpha \rangle$ is not a $\square(\kappa)$-sequence.

Proposition 21. *Suppose* PFA^{++}. *Then every c.c.c. poset* $\mathbb{P}$ *forces* "$\mathsf{WRP}(([\kappa]^\omega)^V, ([\kappa]^\omega)^V)$ *for every* κ".

So $\mathsf{WRP}(S_0, S_1)$ also does not decide 2^ω.

Proposition 22. *Suppose there is a weakly compact cardinal. Then there is a forcing extension in which the following hold:*

① $\mathrm{WRP}([\omega_2]^\omega)$ *holds.*

② $\mathrm{WRP}(S_0, S_1)$ *holds for some stationary* $S_0, S_1 \subseteq [\omega_2]^\omega$.

③ *But* $\mathrm{WRP}^2([\omega_2]^\omega)$ *fails.*

ご清聴ありがとうございました.