Models of some cardinal invariants with large continuum

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Some cardinal invariants

Let X be one of the Polish spaces between 2^ω, ω^ω, $\mathbb{R}$ and $[0,1]$ with the Lebesgue measure. $\mathcal{M}$ denotes the σ-ideal of meager sets of X, $\mathcal{N}$ the σ-ideal of null sets of X. For $\mathcal{I} \in \{\mathcal{M}, \mathcal{N}\}$, let

$\mathrm{add}(\mathcal{I})$ The *additivity of the ideal* $\mathcal{I}$ is the least size of a family $\mathcal{F} \subseteq \mathcal{I}$ which union is not in $\mathcal{I}$.

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General context

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Consider an increasing sequence $\langle\sqsubset_n\rangle_{n<\omega}$ of closed relations in ω^ω and $\sqsubset = \bigcup_{n<\omega} \sqsubset_n$ such that, for every $g \in \omega^\omega$, $\sqsubset^g = \{f \in \omega^\omega \ / \ f \sqsubset g\}$ is meager.

- For a set Y and a real $f \in \omega^\omega$, f *is $\sqsubset$-unbounded over* Y means that $f \not\sqsubset g$ for all $g \in Y \cap \omega^\omega$.
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$\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}_{\pitchfork} \leq \mathrm{non}(\mathcal{M})$ *and* $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{d}_{\pitchfork} \leq \mathrm{non}(\mathcal{N})$.

For $X, A \in [\omega]^\omega$, define

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More cardinal invariants

Say that $\mathcal{F} \subseteq [\omega]^\omega$ is a *filter base* if it is closed under finite intersections and contains all the coinfinite subsets of ω. $A \in [\omega]^\omega$ is a *pseudo-interesection* of $\mathcal{F}$ if $A \subseteq^* X$ for every $X \in \mathcal{F}$. Define

- $\mathfrak{p}$ (pseudo-intersection number): the least size of a filter base without pseudo-intersection.
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Some forcing notions

- $\mathbb{1}$ Trivial forcing $\mathbb{1} = \{0\}$.
- $\mathbb{A}$ Amoeba forcing.
- $\mathbb{B}$ Random forcing.
- $\mathbb{C}$ Cohen forcing.
- $\mathbb{D}$ Hechler forcing.
- $\mathbb{E}$ Eventually different real forcing.

All these are Suslin c.c.c. forcing notions.

- $\mathbb{M}_{\mathcal{F}}$ Mathias forcing with a filter base $\mathcal{F}$.

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- $\mathbb{A}$ Amoeba forcing.
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Preservation properties

Fix κ an uncountable regular cardinal.

For $F \subseteq \omega^\omega$ consider the property

$(\blacktriangle, \sqsubset, F, \kappa)$ For all $X \subseteq \omega^\omega$, if $|X| < \kappa$, then there exists an $f \in F$ which is $\sqsubset$-unbounded over X.

For a forcing notion $\mathbb{P}$, consider the property

$(+^\kappa_{\mathbb{P},\sqsubset})$ $\mathbb{P}$ is κ-c.c. and, for every $\dot{h}$ $\mathbb{P}$-name for a real in ω^ω, there exists a $Y \subseteq \omega^\omega$, $|Y| < \kappa$ such that, for every real f that is $\sqsubset$-unbounded over Y, $\Vdash f \not\sqsubset \dot{h}$.

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$(\blacktriangle, \sqsubset, F, \kappa)$ *implies* $\mathfrak{b}_\sqsubset \leq |F|$ *and* $\kappa \leq \mathfrak{d}_\sqsubset$.

Theorem (Judah and Shelah, 1990, Brendle, 1991)

Forcing notions satisfying $(+^{\kappa}_{\cdot,\sqsubset})$ *preserve* $(\blacktriangle, \sqsubset, F, \kappa)$ *and* $\lambda \leq \mathfrak{d}_\sqsubset$ *for any* $\lambda \geq \kappa$.

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- Every forcing notion of size $< \kappa$ satisfies $(+^{\kappa}_{\cdot,\sqsubset})$. In particular, $(+_{\mathbb{C},\sqsubset})$ holds.
- $(+_{\mathbb{B},<^*})$ and $(+_{\mathbb{E},<^*})$ hold (last by Miller, 1981).
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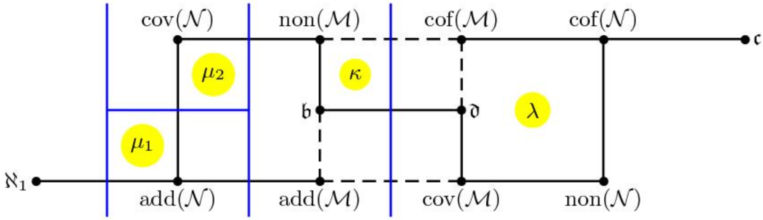
Applications

Theorem

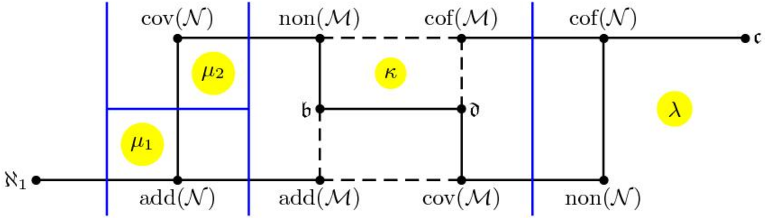
Let $\mu_1 \leq \mu_2 \leq \kappa$ be uncountable regular cardinals, $\lambda \geq \kappa$ a cardinal such that $\mathrm{cf}(\lambda) \geq \kappa$. Then, it is consistent that $\mathrm{add}(\mathcal{N}) = \mu_1$, $\mathrm{cov}(\mathcal{N}) = \mu_2$, $\mathfrak{p} = \mathrm{non}(\mathcal{M}) = \kappa$ and $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \lambda$.



Here, $\mathfrak{s} = \kappa$ and $\mathfrak{r} = \mathfrak{u} = \lambda$.

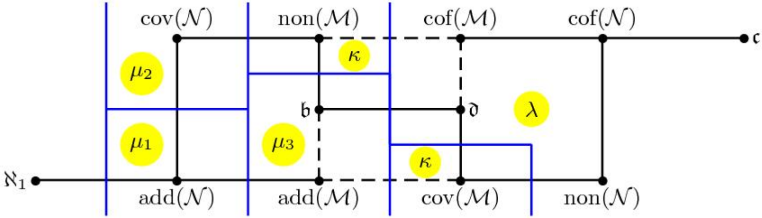
Applications

If $\mu_1 \leq \mu2 \leq \mu_3 \leq \kappa$ are regular uncountable, $\lambda \geq \kappa$ and $\mathrm{cf}(\lambda) \geq \mu_3$, we can get models of ZFC plus:



$\mathfrak{p} = \mathfrak{s} = \mu_3$ and $\mathfrak{r} = \mathfrak{u} = \mathfrak{c} = \lambda$.

Applications



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(2) Which conditions do we require for a suborder $\mathbb{P}$ of $\mathbb{D}$ so that $(+_{\mathbb{P},\in})$ holds?

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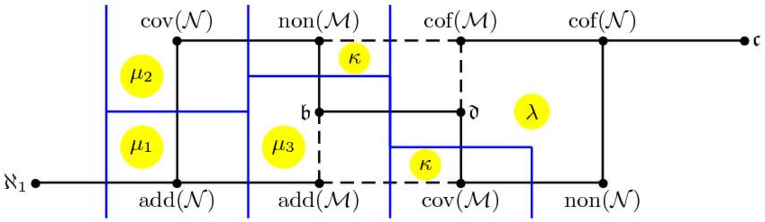

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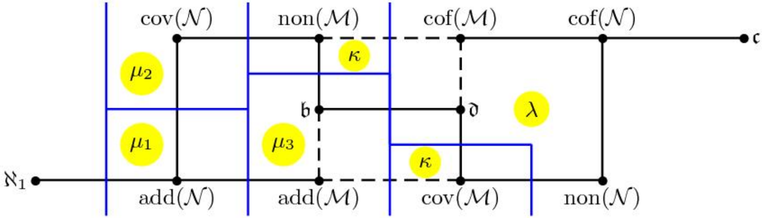

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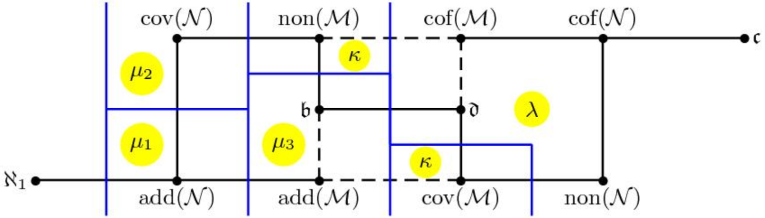

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Relative complete suborder

Fix $M \subseteq N$ transitive standard models of ZFC.

- If $\mathbb{P} \in M$ and $\mathbb{Q}$ are p.o., we say that $\mathbb{P}$ *is a complete suborder of* $\mathbb{Q}$ *respect to* M, denoted by $\mathbb{P} \preceq_M \mathbb{Q}$, iff $\mathbb{P} \subseteq \mathbb{Q}$ and every maximal antichain of $\mathbb{P}$ in M is a maximal antichain of $\mathbb{Q}$.

Theorem (Brendle, Fischer, 2011)

Let δ be an ordinal, $\mathbb{P}_{0,\delta} = \langle \mathbb{P}_{0,\alpha}, \dot{\mathbb{Q}}_{0,\alpha} \rangle_{\alpha<\delta}$ a f.s.i. of c.c.c. forcing defined in M and $\mathbb{P}_{1,\delta} = \langle \mathbb{P}_{1,\alpha}, \dot{\mathbb{Q}}_{1,\alpha} \rangle_{\alpha<\delta}$ a f.s.i. of c.c.c. forcing defined in N. Then, $\mathbb{P}_{0,\delta} \preceq_M \mathbb{P}_{1,\delta}$ iff, for every $\alpha < \delta$, $\mathbb{P}_{0,\alpha} \preceq_M \mathbb{P}_{1,\alpha}$ and $\Vdash_{\mathbb{P}_{1,\alpha},N} \dot{\mathbb{Q}}_{0,\alpha} \preceq_{M^{\mathbb{P}_{0,\alpha}}} \dot{\mathbb{Q}}_{1,\alpha}$.

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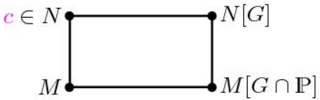
Preservation of ⊏-unbounded reals

Consider the context for the relation $\sqsubset$. If $\mathbb{P} \in M$, $\mathbb{Q} \in N$, $\mathbb{P} \preceq_M \mathbb{Q}$ and $c \in N \cap \omega^\omega$ is a $\sqsubset$-unbounded real over M, define the property

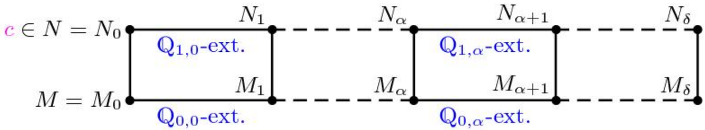
$(\star, \mathbb{P}, \mathbb{Q}, M, N, \sqsubset, c)$ For every $\dot{h} \in M$ $\mathbb{P}$-name for a real in ω^ω, $\Vdash_{\mathbb{Q},N} c \not\sqsubset \dot{h}$. This is equivalent to say that $\Vdash_{\mathbb{Q},N}$ "c is $\sqsubset$-unbounded over $M^{\mathbb{P}}$", i.e., c is $\sqsubset$-unbounded over $M[G \cap \mathbb{P}]$ for every G $\mathbb{Q}$-generic over N.



Preservation of $\sqsubset$-unbounded reals

Theorem (Brendle, Fischer,2011)

With the hypothesis of the previous theorem, if $\mathbb{P}_{0,\delta} \preceq_M \mathbb{P}_{1,\delta}$, $(\star, \mathbb{P}_{0,\delta}, \mathbb{P}_{1,\delta}, M, N, \sqsubset, c)$ *iff, for every* $\alpha < \delta$, $(\star, \mathbb{P}_{0,\alpha}, \mathbb{P}_{1,\alpha}, M, N, \sqsubset, c)$ *and* $\Vdash_{\mathbb{P}_{1,\alpha},N} (\star, \dot{\mathbb{Q}}_{0,\alpha}, \dot{\mathbb{Q}}_{1,\alpha}, M^{\mathbb{P}_{0,\alpha}}, N^{\mathbb{P}_{1,\alpha}}, \sqsubset, c)$.



Cases of preservation of $\sqsubset$-unbounded reals

Theorem

Let $c \in N$ be a $\sqsubset$-unbounded real over M.

(a) *If $\mathbb{P}$ is a Suslin c.c.c. forcing notion with parameters in M and $(+_{\mathbb{P},\sqsubset})$ holds in M, then $(\star, \mathbb{P}^M, \mathbb{P}^N, M, N, \sqsubset, c)$.*

(b) *(Brendle, Fischer, 2011) If $\mathbb{P} \in M$ is a p.o., then $(\star, \mathbb{P}, \mathbb{P}, M, N, \sqsubset, c)$.*

Note also that *every Cohen real over M is $\sqsubset$-unbounded over M.*

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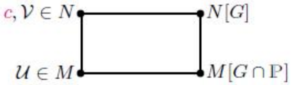
A case of preservation of unbounded reals

Theorem (Blass, Shelah, 1984)

In M, let $\mathcal{U}$ be an ultrafilter. If $c \in N$ is a $<^*$-unbounded real over M, then there exists an ultrafilter $\mathcal{V}$ in N extending $\mathcal{U}$ such that $(\star, \mathbb{M}_{\mathcal{U}}, \mathbb{M}_{\mathcal{V}}, M, N, <^*, c)$ holds.

The same holds if we consider $\subseteq^*$ instead of $<^*$.



Matrix iterations of c.c.c. forcing notions

For δ, γ ordinals, in a ground model V we consider a matrix iteration $\langle\langle\mathbb{P}_{\alpha,\xi}, \dot{\mathbb{Q}}_{\alpha,\xi}\rangle_{\xi<\gamma}\rangle_{\alpha\leq\delta}$ defined by the following conditions.

(1) $\mathbb{P}_{\delta,0} = \langle\mathbb{P}_{\alpha,0}, \dot{\mathbb{R}}_\alpha\rangle_{\alpha<\delta}$ is a f.s.i. of c.c.c. notions.

(2) For a fixed $\alpha \leq \delta$, $\mathbb{P}_{\alpha,\gamma} = \langle\mathbb{P}_{\alpha,\xi}, \dot{\mathbb{Q}}_{\alpha,\xi}\rangle_{\xi<\gamma}$ is a f.s.i. of c.c.c forcing notions.

(3) For $\alpha < \beta \leq \delta, \xi < \gamma$, $\mathbb{P}_{\alpha,\xi} \preceq_V \mathbb{P}_{\beta,\xi}$.

(4) For $\alpha < \beta \leq \delta, \xi < \gamma$, $\Vdash_{\beta,\xi} \dot{\mathbb{Q}}_{\alpha,\xi} \preceq_{V^{\mathbb{P}_{\alpha,\xi}}} \dot{\mathbb{Q}}_{\beta,\xi}$.

(3)+(4) is equivalent to $\mathbb{P}_{\alpha,\xi} \preceq_V \mathbb{P}_{\beta,\xi}$ for every $\alpha \leq \beta \leq \delta$, $\xi \leq \gamma$.

Matrix iterations of c.c.c. forcing notions

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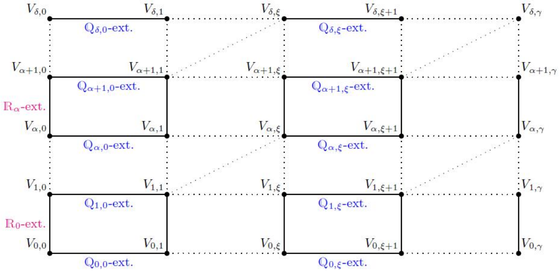
Matrix iterations of c.c.c. forcing notions

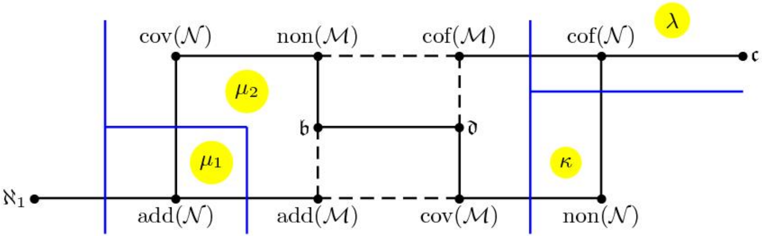
Like in the case of "linear" iterations, $V_{\alpha,\xi}$ denotes a $\mathbb{P}_{\alpha,\xi}$-extension for $\alpha \leq \delta, \xi \leq \gamma$. Here, $V_{0,0} = V$ and the generic extensions can be seen as in the figure.



An application

Theorem

Let $\mu_1 \leq \mu_2 \leq \kappa$ *be uncountable regular cardinals,* $\lambda \geq \kappa$ *a cardinal such that* $\mathrm{cf}(\lambda) \geq \mu_1$*. Then, it is consistent with* ZFC *that* $\mathrm{add}(\mathcal{N}) = \mu_1$, $\mathrm{cov}(\mathcal{N}) = \mathfrak{p} = \mathrm{cof}(\mathcal{M}) = \mu_2$, $\mathrm{non}(\mathcal{N}) = \mathfrak{r} = \kappa$ *and* $\mathrm{cof}(\mathcal{N}) = \mathfrak{c} = \lambda$.



Sketched proof

Start with V a model of ZFC plus $\mathrm{add}(\mathcal{N}) = \mathrm{non}(\mathcal{M}) = \mu_1$ and $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \lambda$. Also, there exists an A of size μ_1 such that $(\blacktriangle, A, \subseteq^*, \mu_1)$. Let $t : \kappa\mu_2 \to \kappa$ such that, for each $\alpha < \kappa$ and $\eta < \kappa\mu_2$, there exists a δ such that $\eta < \delta < \kappa\mu_2$ and $t(\delta) = \alpha$. Also, fix a bijection $g : \lambda \to \kappa \times \lambda$. Perform a matrix iteration $\langle\langle \mathbb{P}_{\alpha,\xi}, \dot{\mathbb{Q}}_{\alpha,\xi}\rangle_{\xi<\lambda\kappa\mu_2}\rangle_{\alpha\leq\kappa}$ (dimensions $\kappa \times (\lambda\kappa\mu_2)$) as follows: let $\mathbb{P}_{\alpha,0}$ be the α-iteration of Cohen forcing, $\dot{c}_\alpha$ the $\mathbb{P}_{\alpha+1,0}$-name of the Cohen real added in the step $\alpha + 1$. We proceed to define the horizontal iterations in the interval $[\lambda\rho, \lambda(\rho + 1))$ for each $\rho < \kappa\mu_2$.

(a) If $\xi = \lambda\rho$, let

$$\dot{\mathbb{Q}}_{\alpha,\xi} = \begin{cases} \mathbb{1}, & \text{if } \alpha \leq t(\rho), \\ \dot{\mathbb{B}}_\rho, & \text{if } \alpha > t(\rho), \end{cases}$$

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(b) If $\xi = \lambda\rho + 1$, $\dot{\mathbb{Q}}_{\alpha,\xi}$ is a $\mathbb{P}_{\alpha,\xi}$-name for $\dot{\mathbb{D}}$.

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Sketched proof

(c) If $\xi = \lambda\rho + 2$, let

$$\dot{\mathbb{Q}}_{\alpha,\xi} = \begin{cases} \mathbb{1}, & \text{if } \alpha \leq t(\rho), \\ \mathbb{M}_{\dot{\mathcal{U}}_\rho}, & \text{if } \alpha > t(\rho), \end{cases}$$

where $\dot{\mathcal{U}}_\rho$ is a $\mathbb{P}_{t(\rho),\xi}$-name for a non-principal ultrafilter on ω.

Now, for $\alpha < \kappa$, consider, $\langle \dot{\mathbb{A}}^\rho_{\alpha,\gamma} \rangle_{\gamma<\lambda}$ and $\langle \dot{\mathcal{F}}^\rho_{\alpha,\gamma} \rangle_{\gamma<\lambda}$ the $\mathbb{P}_{\alpha,\lambda\rho+3}$-names for all suborders of $\mathbb{A}^{V_{\alpha,\lambda\rho+3}}$ of size $< \mu_1$ and all filter basis in $V_{\alpha,\lambda\rho+3}$ of size $< \mu_2$, respectively. For $\epsilon < \lambda$,

(d) If $\xi = \lambda\rho + 3 + 2\epsilon$, put

$$\dot{\mathbb{Q}}_{\alpha,\xi} = \begin{cases} \mathbb{1}, & \text{if } \alpha \leq (g(\epsilon))_0, \\ \dot{\mathbb{A}}^\rho_{g(\epsilon)}, & \text{if } \alpha > (g(\epsilon))_0. \end{cases}$$

(e) If $\xi = \lambda\rho + 3 + 2\epsilon + 1$, put

$$\dot{\mathbb{Q}}_{\alpha,\xi} = \begin{cases} \mathbb{1}, & \text{if } \alpha \leq (g(\epsilon))_0, \\ \mathbb{M}_{\dot{\mathcal{F}}^\rho_{g(\epsilon)}}, & \text{if } \alpha > (g(\epsilon))_0. \end{cases}$$

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(d) If $\xi=\lambda\rho+3+2\epsilon$, put

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Sketched proof

Theorem (Brendle, Fischer, 2011)

If $\xi \leq \lambda\kappa\mu_2$ and x is a real in $V_{\kappa,\xi}$, then $x \in V_{\alpha,\xi}$ for some $\alpha < \kappa$.

In the iterations for each $\rho < \kappa\mu_2$, $\mathbb{B}^{V_{t(\rho),\lambda\rho}}$ adds a random real $r_\rho \in V_{t(\rho)+1,\lambda\rho+1}$ over $V_{t(\rho),\lambda\rho}$ and $\mathbb{M}_{\mathcal{U}_\rho}$ adds a Mathias real $m_\rho \in V_{t(\rho)+1,\lambda\rho+3}$ over $V_{t(\rho),\lambda\rho+2}$.

Claim

For every family of Borel non-null sets coded in $V_{\kappa,\lambda\kappa\mu_2}$ of size $< \mu_2$, there is a r_ρ that is not in its union. Thus, $\mu_2 \leq \mathrm{cov}(\mathcal{N})$ and $\mathrm{non}(\mathcal{N}) \leq \kappa$.

Claim

For every family of size $< \mu_2$ of infinite subsets of ω in $V_{\kappa,\lambda\kappa\mu_2}$ there is some m_ρ which cannot be splitted by any member of the family. Thus, $\mathfrak{r} \leq \kappa$.

Sketched proof

Theorem (Brendle, Fischer, 2011)

If $\xi \leq \lambda\kappa\mu_2$ and x is a real in $V_{\kappa,\xi}$, then $x \in V_{\alpha,\xi}$ for some $\alpha < \kappa$.

In the iterations for each $\rho < \kappa\mu_2$, $\mathbb{B}^{V_{t(\rho),\lambda\rho}}$ adds a random real $r_\rho \in V_{t(\rho)+1,\lambda\rho+1}$ over $V_{t(\rho),\lambda\rho}$ and $\mathbb{M}_{\mathcal{U}_\rho}$ adds a Mathias real $m_\rho \in V_{t(\rho)+1,\lambda\rho+3}$ over $V_{t(\rho),\lambda\rho+2}$.

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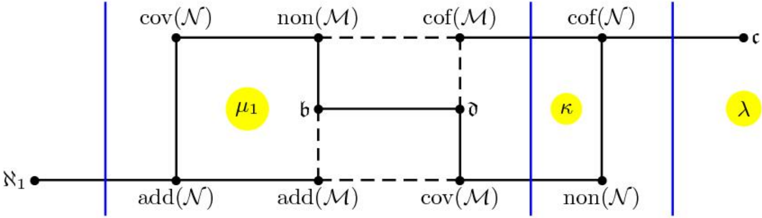
More applications

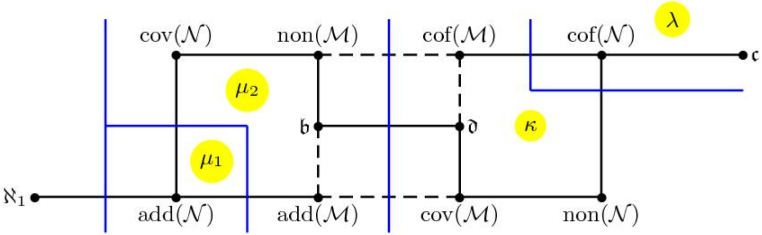
Similarly, with $\mu_1 \leq \mu_2 \leq \mu_3 \leq \kappa$ uncountable regular cardinals, $\lambda \geq \kappa$, we can get models of ZFC plus:
When $\mathrm{cf}(\lambda) \geq \aleph_1$,



Here, $\mathfrak{p} = \mathfrak{s} = \mu_1$ and $\mathfrak{r} = \kappa$.

More applications

When $\mathrm{cf}(\lambda) \geq \mu_1$,



Here, $\mathfrak{p} = \mathfrak{s} = \mu_2$ and $\mathfrak{r} = \kappa$.

More applications

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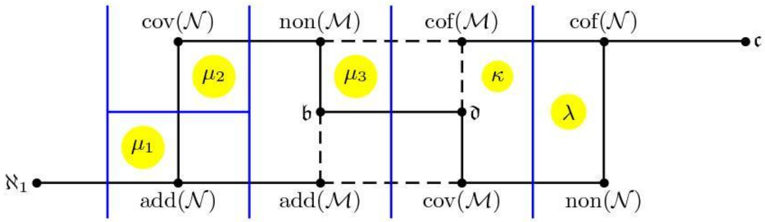

Here, $\mathfrak{p} = \mathfrak{s} = \mu_3$ and $\mathfrak{r} = \kappa$.

More applications

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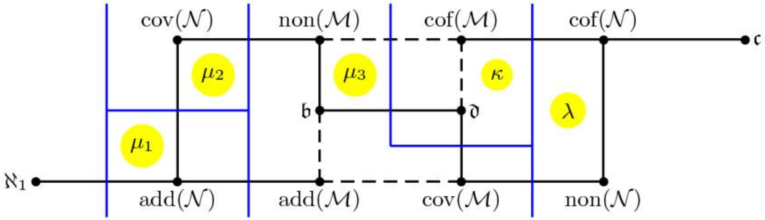


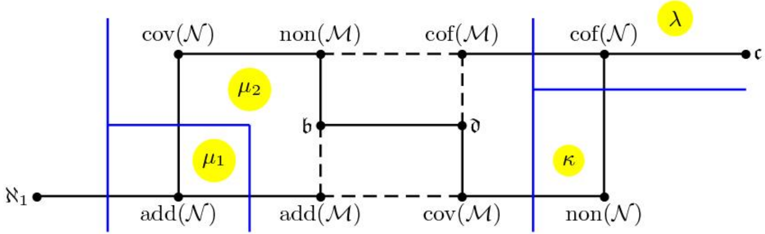
Here, $\mathfrak{p} = \mathfrak{s} = \mathfrak{r} = \mathfrak{u} = \mu_3$.

Questions

Question 2

Does Blass-Shelah Theorem hold for $\pitchfork$ instead of $<^*$?

A positive answer to this will lead to a model of ZFC plus $\mathfrak{u} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) = \mathfrak{c}$.



Questions

Question 3

If $\aleph_1 < \kappa_0 < \kappa_1 < \kappa_2$ for $\kappa_0, \kappa_1, \kappa_2$ regular cardinals, is it consistent with ZFC that $\aleph_1 = \mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) < \kappa_0 = \mathfrak{d} = \mathrm{cof}(\mathcal{M}) < \kappa_1 = \mathrm{non}(\mathcal{N}) < \kappa_2 = \mathrm{cof}(\mathcal{N}) = \mathfrak{c}$?