Ω-logic and Boolean-valued 2nd-order logic

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Ω-logic; Introduction

Most mathematical statements are Π_2 in Set Theory.

Π_2 statements $=$ statements of the form $(\forall\alpha)\ V_\alpha \models \phi$.

In Ω-logic, we focus on the truth of Π_2 statements in Set Theory.

Ω-logic; Ω-validity

Ω-logic: a logic of forcing absoluteness

Definition (Ω-validity)

Let ϕ be a Π_2-sentence in set theory.
Then ϕ is *Ω-valid* if ϕ is true in any set forcing extension.

Main interest: $0^\Omega = \{\phi \mid \phi \text{ is } \Omega\text{-valid}\}$.

Ω-logic; Ω-validity ctd.

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3. (Martin-Solovay) Assume $X^\#$ exists for any set X. Then every Π^1_3-sentence true in V is Ω-valid.
4. (Woodin) If there is a proper class of Woodin cardinals, then for any sentence ϕ true in $\mathrm{L}(\mathbb{R})^V$, $\phi^{\mathrm{L}(\mathbb{R})} \in 0^\Omega$.

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Mouse operators!

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Let $\mathcal{M}_0$ be the transitive collapse of $\mathrm{Hull}_{\Sigma_1}^{\mathcal{M}}(\emptyset)$.

We identify $\mathcal{M}_0$ with $0^\#$.

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Remark

1. If V is closed under the mouse operator $X \mapsto X^{\#}$, then any Π^1_3-sentence true in V is Ω-valid.

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2. If V is closed under the mouse operator $X \mapsto M_\omega^{\#}(X)$, then for any sentence ϕ true in $\mathrm{L}(\mathbb{R})^V$, $\phi^{\mathrm{L}(\mathbb{R})} \in 0^\Omega$.
3. The following are equivalent:
    1. V is closed under the mouse operator $X \mapsto M_\omega^{\#}(X)$, and
    2. $\mathrm{AD}^{\mathrm{L}(\mathbb{R})} \in 0^\Omega$.

Ω-logic; Problems of mouse operators

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Is there any notion in set theory which extracts the properties of mouse operators and which may capture the "essence" of large cardinal properties such as supercompact cardinals in this context?

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One candidate is Universally Baire sets!

Ω-logic; Universally Baire sets

Definition

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A set of reals A is universally Baire if and only if for any partial order $\mathbb{P}$, there are trees T, U on $\omega \times Y$ for some Y such that

$$A = \mathrm{p}[T] \text{ and } \Vdash_{\mathbb{P}} \text{“}\mathrm{p}[\check{T}] = \mathbb{R} \setminus \mathrm{p}[\check{U}]\text{”}.$$

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Example

1. The collection of all uB sets is closed under complements and countable unions, hence every Borel set is universally Baire.
2. Every $\mathbf{\Pi}^1_1$-set of reals is universally Baire.

Ω-logic; Universally Baire sets ctd.

Example

1. The following are equivalent:
   1. every $\mathbf{\Pi}^1_2$-set of reals is universally Baire,
   2. V is closed under the mouse operator $X \mapsto X^\#$, and
   3. $\mathbb{R}$ is closed under the mouse operator $X \mapsto X^\#$, and the function $x \mapsto x^\#$ on the reals is universally Baire.

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2. The following are equivalent:
   1. every set of reals in $\mathrm{L}(\mathbb{R})$ is universally Baire,
   2. V is closed under the mouse operator $X \mapsto M^\#_\omega(X)$, and
   3. $\mathbb{R}$ is closed under the mouse operator $X \mapsto M^\#_\omega(X)$, and the function $x \mapsto M^\#_\omega(x)$ on the reals is universally Baire.

Ω-logic; Closure under universally Baire sets

Definition (A-closure)

Let A be universally Baire. A countable ω-model M of ZFC is *A-closed* if for any M-generic filter G on a partial order in M,

$$M[G] \cap A \in M[G].$$

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2. For a countable ω-model M of ZFC, the following are equivalent:
   1. M is A-closed for every Π^1_2-set A, and
   2. M is closed under the mouse operator $X \mapsto X^{\#}$.

Two beliefs in Berkeley

1. (Mouse set conjecture)

$$\text{Mouse operators} = \text{universally Baire sets}$$

Two beliefs in Berkeley

1. (Mouse set conjecture)

   Mouse operators = universally Baire sets

2. (Ω-conjecture)
   Any phenomenon of forcing absoluteness obtained by strong axioms of infinity must be explained by looking at mouse operators (or uB sets).

Ω-logic; Ω-provability

Definition

Let ϕ be a Π_2-sentence in set theory.
Then ϕ is Ω-provable if there is a universally Baire set A such that

$$(\forall M \text{ c.t.m. of ZFC}) \text{ if } M \text{ is } A\text{-closed, then } M \models \phi.$$

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Example

Suppose V is closed under the mouse operator $X \mapsto X^{\#}$. Then any Π^1_3-sentence true in V is Ω-provable.

Ω-logic; Ω-conjecture

Theorem (Soundness (Woodin))

Let ϕ be a Π_2-sentence. Then ϕ is Ω-provable, then it is Ω-valid.

Ω-logic; Ω-conjecture

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Let ϕ be a Π_2-sentence. Then ϕ is Ω-provable, then it is Ω-valid.

Conjecture (Ω-conjecture (Woodin))

Suppose there is a proper class of Woodin cardinals and let ϕ be a Π_2-sentence. Then ϕ is Ω-provable iff ϕ is Ω-valid.

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Conjecture (Ω-conjecture (Woodin))

Suppose there is a proper class of Woodin cardinals and let ϕ be a Π_2-sentence. Then ϕ is Ω-provable iff ϕ is Ω-valid.

Theorem (Woodin)

ZFC + Ω-conjecture + "There is a proper class of Woodin cardinals" is consistent.

One approach to Ω-conjecture

Question

Does UBH for nice iteration trees hold in any set generic extension?

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If the answer is yes, then

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Does UBH for nice iteration trees hold in any set generic extension?

Theorem

If the answer is yes, then

1. (Woodin) Ω-conjecture holds, and
2. (Asperó-Schindler; Schindler-I.) MM^{++} implies Woodin's Axiom $(*)$ assuming a proper class of Woodin cardinals.

Second order logic; background

Two semantics:

1. Full semantics: Highly complex (very powerful), does not enjoy completeness, ω-compactness.
2. Henkin semantics: Very simple (very week), enjoys completeness, ω-compactness.

Second order logic; background

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1. Full semantics: Highly complex (very powerful), does not enjoy completeness, ω-compactness.
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Boolean valued second order logic is a powerful logic sitting between the two semantics and might enjoy completeness.

2nd-order logic; Henkin models

$$\frac{\text{Henkin models}}{\text{2nd-order logic}} = \frac{\text{Models of ZFC}}{\text{Set theory}}$$

Definition

A 2nd-order structure $M = (X, \mathcal{G}, \ldots)$ is a Henkin model if it satisfies Comprehension Axiom for each 2nd-order formula.

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A 2nd-order structure $M = (X, \mathcal{P}(X), \ldots)$ is called a *full 2nd-order structure*.

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Theorem (Henkin)

The semantics for 2nd-order logic given by Henkin models is sound and complete to a standard proof system in 2nd-order logic.

2nd-order logic; Henkin semantics vs Full semantics

Corollary

The validity of 2nd-order logic via Henkin semantics is Σ^0_1.

Henkin semantics gives us a 2nd-order logic similar to 1st-order logic.

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Full semantics = semantics with full 2nd-order structures

Theorem (Väänänen)

The validity of 2nd-order logic via full semantics is Π_2-complete in ZFC.

Point: One can express the structures of the form $(V_\alpha, \in)$ via full 2nd-order structures.

Boolean valued 2nd-order logic; Boolean valued structures

Definition

Let $\mathcal{L}$ be a relational language. A Boolean valued $\mathcal{L}$-structure is a tuple $M = (A, \mathbb{B}, \{R_i^M\})$ where

1. A is a nonempty set,
2. $\mathbb{B}$ is a complete Boolean algebra, and
3. for each n-ary relational symbol R_i in $\mathcal{L}$, $R_i^M \colon A^n \to \mathbb{B}$.

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2. $\mathbb{B}$ is a complete Boolean algebra, and
3. for each n-ary relational symbol R_i in $\mathcal{L}$, $R_i^M : A^n \to \mathbb{B}$.

Example

If $\mathbb{B} = \{0, 1\}$, each R_i^M is a relation in 1st-order logic and M is the same as 1st-order structure.

Truth of 2nd-order formulas in Boolean valued structures

Basic idea: "subsets" are functions from A to $\mathbb{B}$.

Definition

Let $M = (A, \mathbb{B}, \{R_i\})$ be a Boolean valued $\mathcal{L}$-structure. Then we assign $\|\phi[\vec{a}, \vec{f}]\|^M \in \mathbb{B}$ to each 2nd-order formula ϕ, $\vec{a} \in {}^{<\omega}A$, and $\vec{f} \in {}^{<\omega}({}^A\mathbb{B})$ as follows:

1. ϕ is $R_i(\vec{x})$. Then $\|R_i(\vec{x})[\vec{a}]\|^M = R_i^M(\vec{a})$.
2. ϕ is $X(x)$. Then $\|X(x)[a, f]\|^M = f(a)$.
3. Boolean combinations are as usual.
4. ϕ is $\exists x\psi$. Then $\|\exists x\psi[\vec{a}, \vec{f}]\|^M = \bigvee_{b \in A} \|\psi[b, \vec{a}, \vec{f}]\|^M$.
5. ϕ is $\exists X\psi$. Then $\|\exists X\psi[\vec{a}, \vec{f}]\|^M = \bigvee_{g\,:\,A \to \mathbb{B}} \|\psi[\vec{a}, g, \vec{f}]\|^M$.

Boolean valued 2nd-order logic; Boolean-validity

Definition

Let $\mathcal{L}$ be relational. A 2nd-order $\mathcal{L}$-sentence ϕ is Boolean-valid if $\|\phi\|^M = 1$ for any Boolean valued $\mathcal{L}$-structure M.

Our interest: $0^{2b} = \{\phi \mid \phi \text{ is Boolean-valid}\}$.

Result 1; Validity

Theorem

$$0^{\Omega} \equiv_T 0^{2b}.$$

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Theorem (Woodin)

Assuming Ω-conjecture and a proper class of Woodins, one can show that 0^Ω is Δ_2 in Set Theory.

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Corollary

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Remark (Väänänen)

The validity of second order logic via full semantics is Π_2-complete in ZFC.

Result 1; Validity ctd.

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Theorem

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Key points:

Remark

Given a Boolean valued structure $M = (A, \mathbb{B}, \{R_i^M\})$ and a $\mathbb{B}$-generic filter G over V, the structure M corresponds to a full 2nd-order structure $M[G] = \left(A, \mathcal{P}(A)^{V[G]}, \{R_i^{M[G]}\}\right)$ in $V[G]$, where

$$R_i^{M[G]} = \{x \in A, | \ R_i^M(x) \in G\}.$$

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$$R_i^{M[G]} = \{x \in A, | \ R_i^M(x) \in G\}.$$

For any 2nd-order sentence ϕ, $\|\phi\|^M = 1$ iff $M[G] \models \phi$ for any $\mathbb{B}$-generic filter over V.

Result 2; Compactness numbers

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The following are equivalent:

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Theorem (Magidor)

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2. κ is extendible.

Theorem

Suppose there is a proper class of Woodin cardinals, a supercompact cardinal κ, and assume Strong Ω-conjecture holds.
Then κ is $\mathrm{L}^{2b}_{\kappa,\kappa}$-compact.

Definition (Strong Ω-conjecture)

Assume there is a proper class of Woodin cardinals. Then Ω-conjecture with real parameters holds in any set generic extension.

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Given a logic L, the Löwenheim-Skolem number of L ($\ell\,(L)$) is the least κ such that

$$(\forall \phi \in L)\ (\exists M)\ M \vDash \phi \implies (\exists M)\ M \vDash \phi \text{ and } \mathrm{card}(M) \leq \kappa.$$

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Example

1. $\ell(\mathrm{FOL}) = \aleph_0$

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Definition

Given a logic L, the Löwenheim-Skolem number of L ($\ell(L)$) is the least κ such that

$$(\forall \phi \in L)\ (\exists M)\ M \models \phi \implies (\exists M)\ M \models \phi \text{ and } \mathrm{card}(M) \leq \kappa.$$

Example

1. $\ell(\mathrm{FOL}) = \aleph_0$
2. $\ell(\text{full SOL}) = \sup\{\alpha \mid \alpha \text{ is } \Delta_2\text{-definable}\}$. So

$$(\text{The first Woodin limit of Woodins}) < \ell(\text{full SOL}) \leq (\text{The first } \Sigma_2 \text{ reflecting card}).$$

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Theorem

If ZFC + "There is a proper class of Woodin cardinals" is consistent, then so is ZFC + "There is a proper class of Woodin cardinals" + "ℓ (BVSOL) < (the first Woodin cardinal)"

Result 4; Completeness

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Assume there is a proper class of Woodin cardinals. Then if ϕ is Boolean valid, then so is Boolean provable.

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Completeness of BVSOL implies Ω-conjecture.

Note: The converse is not known to be true.

Questions

1. Does Ω-conjecture imply the Completeness of BVSOL?
2. Could ℓ (BVSOL) be less than the first measurable cardinal?