Generic setwise large cardinals

Toshimichi Usuba (薄葉季路)

Waseda University (早稲田大学)

March 10, 2021
Kobe Set Theory Workshop 2021
–on the occasion of Sakaé Fuchino's retirement–

Generic large cardinal

- *Generic large cardinal* is defined by the existence of *generic elementary embedding*: An elementary embedding which is living in some generic extension.

Definition

A cardinal κ is *generically measurable* if there is a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there are a transitive class and an elementary embedding $j : V \to M$ with critical point κ (M and j may not be definable in V).

- Unlike usual large cardinals, generic large cardinals can be small.

Remark

1. If κ is measurable and $\delta < \kappa$ is regular, then $\mathrm{Col}(\delta, < \kappa)$ forces $\kappa = \delta^+$ and κ is generically measurable. Hence ω_1 can be generically measurable.
2. However its consistency strength is not weak: CON($\exists$generically measurable) $\iff$ CON($\exists$measurable).

- Unlike usual large cardinals, generic large cardinals can be small.

Remark

1. If κ is measurable and $\delta < \kappa$ is regular, then $\mathrm{Col}(\delta, < \kappa)$ forces $\kappa = \delta^+$ and κ is generically measurable. Hence ω_1 can be generically measurable.
2. However its consistency strength is not weak: CON($\exists$generically measurable) $\Longleftrightarrow$ CON($\exists$measurable).

Setwise large cardinals

- Usually the source model of an elementary embedding is supposed to be a proper class V.
- However, the source model of an elementary embedding can be arbitrary large sets in many cases.

Observation

A cardinal κ is supercompact if and only if it is *setwise supercompact*: For every regular λ, there is a transitive set N with ${}^{\lambda}N \subseteq N$ and an elementary embedding $j : H_\lambda \to N$ with critical point κ and $\lambda < j(\kappa)$.

Generic large cardinals

Nielsen and Schlicht introduced restricted generic large cardinals, the source model of a generic elementary embedding is a *set living in V*.

Definition (Nielsen-Schlicht)

A cardinal κ is *generically setwise supercompact* if for every regular $\lambda \geq \kappa$, there is a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there is a transitive set N and an elementary embedding $j : H_\lambda^V \to N$ with critical point κ, $\lambda < j(\kappa)$, and ${}^\lambda N \subseteq N$ (j and N may not be in V).

Setwise extendible

An uncountable cardinal κ is *extendible* if for every $\alpha \geq \kappa$, there are $\beta > \alpha$ and an elementary embedding $j : V_\alpha \to V_\beta$ with critical point κ and $\alpha < j(\kappa)$. Extendible cardinal is originally setwise large cardinal.

Definition (Nielsen-Schlicht)

A cardinal κ is *generically extendible* if for every $\alpha > \kappa$, there is a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there are $\beta > \alpha$ and an elementary embedding $j : V_\alpha \to V_\beta^{\mathbb{P}}$ with critical point κ and $\alpha < j(\kappa)$.

Extendible cardinals can be characterized by the compacteness number of second-order infinitary logic (Magidor).

Theorem (Ikegami-Väänänen)

The compactness number of Boolean second-order logic equals the first generically extendible cardinal.

Setwise extendible

An uncountable cardinal κ is *extendible* if for every $\alpha \geq \kappa$, there are $\beta > \alpha$ and an elementary embedding $j : V_\alpha \to V_\beta$ with critical point κ and $\alpha < j(\kappa)$. Extendible cardinal is originally setwise large cardinal.

Definition (Nielsen-Schlicht)

A cardinal κ is *generically extendible* if for every $\alpha > \kappa$, there is a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there are $\beta > \alpha$ and an elementary embedding $j : V_\alpha \to V_\beta^{\mathbb{P}}$ with critical point κ and $\alpha < j(\kappa)$.

Extendible cardinals can be characterized by the compacteness number of second-order infinitary logic (Magidor).

Theorem (Ikegami-Väänänen)

The compactness number of Boolean second-order logic equals the first generically extendible cardinal.

Setwise extendible

An uncountable cardinal κ is *extendible* if for every $\alpha \geq \kappa$, there are $\beta > \alpha$ and an elementary embedding $j : V_\alpha \to V_\beta$ with critical point κ and $\alpha < j(\kappa)$. Extendible cardinal is originally setwise large cardinal.

Definition (Nielsen-Schlicht)

A cardinal κ is *generically extendible* if for every $\alpha > \kappa$, there is a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there are $\beta > \alpha$ and an elementary embedding $j : V_\alpha \to V_\beta^{\mathbb{P}}$ with critical point κ and $\alpha < j(\kappa)$.

Extendible cardinals can be characterized by the compacteness number of second-order infinitary logic (Magidor).

Theorem (Ikegami-Väänänen)

The compactness number of Boolean second-order logic equals the first generically extendible cardinal.

Setwise extendible

An uncountable cardinal κ is *extendible* if for every $\alpha \geq \kappa$, there are $\beta > \alpha$ and an elementary embedding $j : V_\alpha \to V_\beta$ with critical point κ and $\alpha < j(\kappa)$. Extendible cardinal is originally setwise large cardinal.

Definition (Nielsen-Schlicht)

A cardinal κ is *generically extendible* if for every $\alpha > \kappa$, there is a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there are $\beta > \alpha$ and an elementary embedding $j : V_\alpha \to V_\beta^{\mathbb{P}}$ with critical point κ and $\alpha < j(\kappa)$.

Extendible cardinals can be characterized by the compacteness number of second-order infinitary logic (Magidor).

Theorem (Ikegami-Väänänen)

The compactness number of Boolean second-order logic equals the first generically extendible cardinal.

- Extendible $\Rightarrow$ generically extendible.
- Supercompact $\Rightarrow$ Generically setwise supercompact.

Observation

1. If there are proper class many Woodin cardinals, then every regular uncountable cardinal is generically extendible (via stationary tower forcing).
2. Extendible and supercompact cardinal have very strong consistency strengths, but the consistency of generic version is weaker than the proper class of Woodin cardinals.

Question

What are the consistency strengths of "$\exists$ generically setwise supercompact" and "$\exists$ generically extendible"?

- Extendible ⇒ generically extendible.
- Supercompact ⇒ Generically setwise supercompact.

Observation

1. If there are proper class many Woodin cardinals, then every regular uncountable cardinal is generically extendible (via stationary tower forcing).
2. Extendible and supercompact cardinal have very strong consistency strengths, but the consistency of generic version is weaker than the proper class of Woodin cardinals.

Question

What are the consistency strengths of "∃ generically setwise supercompact" and "∃ generically extendible"?

- Extendible ⇒ generically extendible.
- Supercompact ⇒ Generically setwise supercompact.

Observation

1. If there are proper class many Woodin cardinals, then every regular uncountable cardinal is generically extendible (via stationary tower forcing).
2. Extendible and supercompact cardinal have very strong consistency strengths, but the consistency of generic version is weaker than the proper class of Woodin cardinals.

Question

What are the consistency strengths of "∃ generically setwise supercompact" and "∃ generically extendible"?

- Extendible ⇒ generically extendible.
- Supercompact ⇒ Generically setwise supercompact.

Observation

1. If there are proper class many Woodin cardinals, then every regular uncountable cardinal is generically extendible (via stationary tower forcing).
2. Extendible and supercompact cardinal have very strong consistency strengths, but the consistency of generic version is weaker than the proper class of Woodin cardinals.

Question

What are the consistency strengths of "$\exists$ generically setwise supercompact" and "$\exists$ generically extendible"?

Main result

- The consistency strength of generically extendible cardinal is very weak.

Theorem (U.)

"ω_1 is generically extendible" is equiconsistent with some weak large cardinal axiom, which coexists with $V = L$, and is implied by $0^{\#}$.

Main result

- The consistency strength of generically extendible cardinal is very weak.

Theorem (U.)

"ω_1 is generically extendible" is equiconsistent with some weak large cardinal axiom, which coexists with $V = L$, and is implied by $0^{\#}$.

Virtual large cardinals

Gitman and Schindler introduced the notion of *virtual large cardinals*: It is a variant of generic large cardinal, but the source model and the target model are *sets living in V*.

Definition (Gitman-Schindler)

A cardinal κ is *virtually extendible* if for every $\alpha > \kappa$, there are $\beta > \alpha$ and a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there is an elementary embedding $j : V_\alpha \to V_\beta$ with critical point κ (j may not be in V).

Theorem (Gitman-Schindler)

1. If $0^\#$ exists, then every Silver indiscernible is virtually extendible in L. In particular, every uncountable cardinal is virtually extendible in L.
2. If κ is virtually extendible, then so is in L.

Virtual large cardinals

Gitman and Schindler introduced the notion of *virtual large cardinals*: It is a variant of generic large cardinal, but the source model and the target model are *sets living in V*.

Definition (Gitman-Schindler)

A cardinal κ is *virtually extendible* if for every $\alpha > \kappa$, there are $\beta > \alpha$ and a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there is an elementary embedding $j : V_\alpha \to V_\beta$ with critical point κ (j may not be in V).

Theorem (Gitman-Schindler)

1. If $0^\#$ exists, then every Silver indiscernible is virtually extendible in L. In particular, every uncountable cardinal is virtually extendible in L.
2. If κ is virtually extendible, then so is in L.

Theorem (Gitman-Schindler)

Let M be an inner model, and $X, Y \in M$ transitive sets. Let $j : X \to Y$ be an elementary embedding (j may not be in M) with critical point δ, and $a \subseteq X$ a finite set. Then the forcing $\mathrm{Col}(|X|^M)$ over M forces that "there is an elementary embedding $i : X \to Y$ with critical point δ and $i(x) = j(x)$ for every $x \in a$".

Take a $(V, \mathrm{Col}(|X|^M))$-generic G, and fix an enumeration $\{x_n \mid n < \omega\} \in M[G]$ of X. Now let T be the set of all finite partial elementary embedding i from X to Y such that:

- the the critical point of i is δ.
- $\mathrm{dom}(i) = \{x_n \mid n < |i|\}$ and $i(x) = j(x)$ for every $x \in a \cap \mathrm{dom}(i)$.

T with the initial segment-end extension relation forms a tree. We know $T \in M[G]$.

j witnesses that T is ill-founded in $V[G]$. Hence T is also ill-founded in $M[G]$, and we can take a cofinal branch $f \in M[G]$ of T. f generates a required elementary embedding.

Virtual is generic, Generic is virtual

Theorem (U.)

1. If κ is generically extendible, then κ is virtually extendible in L.
2. If κ is virtually extendible, then $\mathrm{Col}(\omega, <\kappa)$ forces that "$\kappa = \omega_1$ is generically extendible".

Corollary

$$\begin{aligned} \mathrm{CON}(\exists 0^{\#}) &\Longrightarrow \mathrm{CON}(\exists \text{ virtually extendible}) \\ &\Longleftrightarrow \mathrm{CON}(\exists \text{ generically extendible}) \\ &\Longleftrightarrow \mathrm{CON}(\omega_1 \text{ is generically extendible}). \end{aligned}$$

Virtual is generic, Generic is virtual

Theorem (U.)

1. If κ is generically extendible, then κ is virtually extendible in L.
2. If κ is virtually extendible, then $\mathrm{Col}(\omega, <\kappa)$ forces that "$\kappa = \omega_1$ is generically extendible".

Corollary

$$\begin{aligned} \mathsf{CON}(\exists 0^{\#}) &\Longrightarrow \mathsf{CON}(\exists \text{ virtually extendible}) \\ &\Longleftrightarrow \mathsf{CON}(\exists \text{ generically extendible}) \\ &\Longleftrightarrow \mathsf{CON}(\omega_1 \text{ is generically extendible}). \end{aligned}$$

Sketch of the proof

1. If κ is generically extendible, then κ is virtually extendible in L.
2. If κ is virtually extendible, then $\mathrm{Col}(\omega, <\kappa)$ forces that $\kappa = \omega_1$ is generically extendible.

(1). Fix $\alpha > \kappa$, and take a poset $\mathbb{P}$ which forces that "there is an elementary embedding $j : V_\alpha \to V[G]_\beta$ for some β". $j \restriction V_\alpha^L$ is an elementary embedding from V_α^L to V_β^L. By Gitman-Schindler's theorem, there is an elementary embedding $i : V_\alpha^L \to V_\beta^L$ in $L^{\mathrm{Col}(|V_\alpha^L|)}$.

Sketch of the proof

1. If κ is generically extendible, then κ is virtually extendible in L.
2. If κ is virtually extendible, then $\mathrm{Col}(\omega, <\kappa)$ forces that $\kappa = \omega_1$ is generically extendible.

(2). Take a $(V, \mathrm{Col}(\omega, <\kappa))$-generic G.
Fix a large $\alpha > \kappa$. By Gitman-Schindler's theorem, $\mathrm{Col}(|V_\alpha|)$ forces that there is an elementary embedding $j : V_\alpha \to V_\beta$ and $\alpha < j(\kappa)$.
Take a $(V, \mathrm{Col}(\omega, < j(\kappa))$-generic H extending G. In $V[H](= V[G][H])$, there is an elementary embedding $j : V_\alpha \to V_\beta$, and j can be extended to $j : V_\alpha[G] \to V_\beta[H]$. Since $V_\alpha[G] = V[G]_\alpha$ and $V_\beta[H] = V[H]_\beta$, j is an elementary embedding from $V[G]_\alpha$ to $V[G][H]_\beta$, that is, in $V[G]$, $\mathrm{Col}(\omega, < j(\kappa))$ forces that "there is an elementary embedding from $V[G]_\alpha$ to $V[G][H]_\beta$".

Indestructibility of generic setwise large cardinals

Theorem

1. (Laver) After some preparation forcing, a supercompact cardinal κ is indestructible under $< \kappa$-directed closed forcing.
2. (Bagaria-Hamkins-Tsaprounis-U.) If κ is extendible, then every non-trivial $< \kappa$-closed forcing destroys the extendibility of κ.

Theorem (Nielsen-Schlicht, U.)

Suppose κ is generically extendible (generically setwise supercompact, resp.)

1. Every κ-c.c. forcing preserves the generic extendibility (generic setwise supercompactness, resp.) of κ.
2. If $\kappa = \omega_1$, then every proper forcing preserves the generic extendibility (generic setwise supercompactness, resp.) of κ.

Indestructibility of generic setwise large cardinals

Theorem

1. (Laver) After some preparation forcing, a supercompact cardinal κ is indestructible under $< \kappa$-directed closed forcing.
2. (Bagaria-Hamkins-Tsaprounis-U.) If κ is extendible, then every non-trivial $< \kappa$-closed forcing destroys the extendibility of κ.

Theorem (Nielsen-Schlicht, U.)

Suppose κ is generically extendible (generically setwise supercompact, resp.)

1. Every κ-c.c. forcing preserves the generic extendibility (generic setwise supercompactness, resp.) of κ.
2. If $\kappa = \omega_1$, then every proper forcing preserves the generic extendibility (generic setwise supercompactness, resp.) of κ.

Remark

1. If there are proper class many Woodin cardinals, then every regular uncountable cardinal is generically extendible.
2. So, for a given regular uncountable cardinal κ and a poset $\mathbb{P}$, if $\mathbb{P}$ preserves the regularity of κ then $\mathbb{P}$ also preserves the generic extendibility of κ.

Question (Nielsen-Schlicht)

1. Even if $\kappa > \omega_1$, is the generic extendibility (generic setwise supercompactness, resp.) of κ preserved by κ-directed closed forcing?
2. Is it consistent that there is a poset $\mathbb{P}$ which preserves the regularity of κ but destroys the generic extendibility (generic setwise supercompactness, resp.)?

Remark

1. If there are proper class many Woodin cardinals, then every regular uncountable cardinal is generically extendible.
2. So, for a given regular uncountable cardinal κ and a poset $\mathbb{P}$, if $\mathbb{P}$ preserves the regularity of κ then $\mathbb{P}$ also preserves the generic extendibility of κ.

Question (Nielsen-Schlicht)

1. Even if $\kappa > \omega_1$, is the generic extendibility (generic setwise supercompactness, resp.) of κ preserved by κ-directed closed forcing?
2. Is it consistent that there is a poset $\mathbb{P}$ which preserves the regularity of κ but destroys the generic extendibility (generic setwise supercompactness, resp.)?

Generically setwise supercompact $> \omega_1$

- The generic setwise large cardinal property of ω_1 has a spacial place: Consistency is weak, it is always indestructible by certain forcing.
- How about cardinals $> \omega_1$?

Question (Nielsen-Schlicht)

Does the existence of generic setwise supercompact cardinal $> \omega_1$ imply $0^\#$?

Theorem (U.)

If there is a generically setwise supercompact cardinal $> \omega_2$, then $0^\#$ exists.

Generically setwise supercompact $> \omega_1$

- The generic setwise large cardinal property of ω_1 has a spacial place: Consistency is weak, it is always indestructible by certain forcing.
- How about cardinals $> \omega_1$?

Question (Nielsen-Schlicht)

Does the existence of generic setwise supercompact cardinal $> \omega_1$ imply $0^{\#}$?

Theorem (U.)

If there is a generically setwise supercompact cardinal $> \omega_2$, then $0^{\#}$ exists.

Lemma (Folklore?)

Suppose there is an elementary embedding $j : L_\alpha \to L_\beta$ with critical point δ. If $\omega_2 \leq \delta < (\delta^+)^L \leq \alpha$, then $0^\#$ exists.

Suppose $0^\#$ does not exist. Let $U = \{X \in \mathcal{P}(\delta)^L \mid \delta \in j(X)\}$. U is an L-ultrafilter over δ. Then the ultrapower of L by U is well-founded: If not, we can find functions $\{f_n \mid n < \omega\} \subseteq L$ witness the ill-foundedness of the ultrapower. By Jensen's covering lemma, we can find $X \in L$ such that the size of X is ω_1 and $\{f_n \mid n < \omega\} \subseteq X$. Because $\omega_2 \leq \delta$, we have $|X|^L < \delta$. Then the rest follows from the standard condensation argument.

If there is a generically setwise supercompact cardinal $> \omega_2$, then $0^\#$ exists.

Fix a large λ, and take a poset $\mathbb{P}$ which forces that "there are N and an elementary embedding $j : H_\lambda^V \to N$ with critical point $> \omega_2^V$ and ${}^\lambda N \subseteq N$".
In $V^{\mathbb{P}}$, since the critical point of j is $> \omega_2^V$ we have
$\omega_1^V = j(\omega_1^V) = \omega_1^N = \omega_1^{V^{\mathbb{P}}}$ and $\omega_2^V = j(\omega_2^V) = \omega_2^{V^{\mathbb{P}}}$, hence ω_1 and ω_2 are preserved. $j \restriction L_\lambda$ is an elementary embedding from L_λ to L_β. By the lemma above, we have that $0^\#$ exists.

By the covering lemma of Dodd-Jensen core model, we also have:

Theorem (U.)

CON($\exists$ generically setwise supercompact cardinal $> \omega_2$)
$\Longrightarrow$ CON($\exists$ measurable cardinal).

If there is a generically setwise supercompact cardinal $> \omega_2$, then $0^\#$ exists.

Fix a large λ, and take a poset $\mathbb{P}$ which forces that "there are N and an elementary embedding $j : H_\lambda^V \to N$ with critical point $> \omega_2^V$ and ${}^\lambda N \subseteq N$".
In $V^\mathbb{P}$, since the critical point of j is $> \omega_2^V$ we have $\omega_1^V = j(\omega_1^V) = \omega_1^N = \omega_1^{V^\mathbb{P}}$ and $\omega_2^V = j(\omega_2^V) = \omega_2^{V^\mathbb{P}}$, hence ω_1 and ω_2 are preserved. $j \restriction L_\lambda$ is an elementary embedding from L_λ to L_β. By the lemma above, we have that $0^\#$ exists.

By the covering lemma of Dodd-Jensen core model, we also have:

Theorem (U.)

CON($\exists$ generically setwise supercompact cardinal $> \omega_2$)
$\Longrightarrow$ CON($\exists$ measurable cardinal).

Question

What is the exact consistency strength of the generic extendibility (generic setwise supercompactness) of a cardinal $> \omega_2$?

- An upper bound is a proper class of Woodin cardinals, a lower is a measurable cardinal.

What about ω_2?

Question

What is the consistency strength of the generic extendibility (generic setwise supercompactness) of ω_2?

Theorem (U.)

Suppose GCH, and κ is virtually extendible. Then there is a forcing extension $V^{\mathbb{P}}$ in which the following hold:

1. $\kappa = \omega_2$ and $\omega_1 = \omega_1^V$.
2. For every regular $\lambda < \aleph_{\omega_1}$, there is a poset $\mathbb{Q}$ which forces that: There are a regular θ and an elementary embedding $j : H_\lambda^{V^{\mathbb{P}}} \to H_\theta^{V^{\mathbb{P}*\mathbb{Q}}}$ with critical point κ.

What about ω_2?

Question

What is the consistency strength of the generic extendibility (generic setwise supercompactness) of ω_2?

Theorem (U.)

Suppose GCH, and κ is virtually extendible. Then there is a forcing extension $V^{\mathbb{P}}$ in which the following hold:

1. $\kappa = \omega_2$ and $\omega_1 = \omega_1^V$.
2. For every regular $\lambda < \aleph_{\omega_1}$, there is a poset $\mathbb{Q}$ which forces that: There are a regular θ and an elementary embedding $j : H_\lambda^{V^{\mathbb{P}}} \to H_\theta^{V^{\mathbb{P}*\mathbb{Q}}}$ with critical point κ.

Sketch of the proof

1. Use Jensen's poset $\mathbb{P}$: $\mathbb{P}$ has the κ-c.c., and $\mathbb{P}$ forces that $\kappa = \omega_2$, $\omega_1 = \omega_1^V$, and every regular cardinal λ in V with $\omega_1 < \lambda < \kappa$ has cofinality ω.
2. Fix a regular $\lambda < \kappa^{+\omega_1}$. In $V^{\mathrm{Col}(\lambda)}$, there is an elementary embedding $j : H_\lambda^V \to H_\theta^V$ with critical point κ. $\mathbb{P}$ is a complete suborder of $j(\mathbb{P})$.
3. Fix $(V^{\mathrm{Col}(\lambda)}, j(\mathbb{P}))$-generic H, and $G = j^{-1}``H$, this is $(V, \mathbb{P})$-generic.
4. j can be extended to $j : H_\lambda^{V[G]} \to H_\theta^{V[H]}$.
5. $\mathrm{cf}(\lambda) = \omega$ in $V[H]$. Since λ is small, in $V[H]$ we can find $\{X_n \mid n < \omega\}$ such that $X_n \in V[G]$, $|X_n|^{V[G]} \leq \omega_1$, and $\bigcup_n X_n = H_\lambda^{V[G]}$.
6. j may not be in $V[H]$, but we have $j \restriction X_n \in V[H]$.
7. By a variant of Gitman-Schindler's theorem, in $V[H]$ we can find an elementary embedding $i : H_\lambda^{V[G]} \to H_\theta^{V[H]}$ with critical point κ.

Question

Does the Jensen's poset force the following?: For every cardinal λ in V with $\omega_1 < \lambda < \kappa$, there is a family $\{X_n \mid n < \omega\}$ such that $X_n \in V$, $|X_n|^V \leq \omega_1$, and $\lambda = \bigcup_n X_n$.

- It is O.K. if $\lambda < \aleph_{\omega_1}$.
- If the answer is "yes", then we can prove that "ω_2 is generically extendible" is equiconsistent with "there is a virtually extendible".

Generically supercompact v.s. Generically extendible

- Every extendible cardinal is a limit of supercompact cardinals. In particular, the consistency strength of extendible cardinal is much stronger than supercompact.
- How about generic extendible and generic supercompact?

Theorem (U.)

If κ is generically setwise supercompact, then κ is virtually extendible in L.

Corollary

CON($\exists$ generically setwise supercompact)
$\iff$ CON($\exists$ generically extendible)

Generically supercompact v.s. Generically extendible

- Every extendible cardinal is a limit of supercompact cardinals. In particular, the consistency strength of extendible cardinal is much stronger than supercompact.
- How about generic extendible and generic supercompact?

Theorem (U.)

If κ is generically setwise supercompact, then κ is virtually extendible in L.

Corollary

CON($\exists$ generically setwise supercompact)
$\iff$ CON($\exists$ generically extendible)

Generically supercompact v.s. Generically extendible

- Every extendible cardinal is a limit of supercompact cardinals. In particular, the consistency strength of extendible cardinal is much stronger than supercompact.
- How about generic extendible and generic supercompact?

Theorem (U.)

If κ is generically setwise supercompact, then κ is virtually extendible in L.

Corollary

CON($\exists$ generically setwise supercompact)
$\iff$ CON($\exists$ generically extendible)

Jointly large cardinals

Theorem (Tsaprounis)

κ is extendible if and only if κ is jointly supercompact and superstrong: For every $\alpha > \kappa$, there is an elementary embedding $j : V \to M$ such that the critical point of j is κ, $\alpha < j(\kappa)$, ${}^{\alpha}M \subseteq M$, and $V_{j(\kappa)} \subseteq M$.

Proposition

κ is virtually extendible if and only if κ is virtually jointly setwise supercompact and superstrong: For every $\alpha > \kappa$, there is a transitive set N and a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$: there is an elementary embedding $j : V_\alpha \to N$ with critical point κ, $\alpha < j(\kappa)$, ${}^{\alpha}N \cap V \subseteq N$, and $V_{j(\kappa)} \subseteq N$. Moreover it is equivalent to virtually jointly setwise strong and superstrong.

Jointly large cardinals

Theorem (Tsaprounis)

κ is extendible if and only if κ is jointly supercompact and superstrong: For every $\alpha > \kappa$, there is an elementary embedding $j : V \to M$ such that the critical point of j is κ, $\alpha < j(\kappa)$, ${}^{\alpha}M \subseteq M$, and $V_{j(\kappa)} \subseteq M$.

Proposition

κ is virtually extendible if and only if κ is virtually jointly setwise supercompact and superstrong: For every $\alpha > \kappa$, there is a transitive set N and a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$: there is an elementary embedding $j : V_{\alpha} \to N$ with critical point κ, $\alpha < j(\kappa)$, ${}^{\alpha}N \cap V \subseteq N$, and $V_{j(\kappa)} \subseteq N$. Moreover it is equivalent to virtually jointly setwise strong and superstrong.

If κ is generically setwise supercompact, then κ is virtually extendible in L.

We see that κ is virtually jointly setwise supercompact and superstrong in L. If $\kappa > \omega_2$, then $0^{\#}$ exists and it is O.K. If $\kappa \leq \omega_2$, then κ is successor, say $\kappa = \mu^+$.

Fix $\alpha > \kappa$, and take a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there are a transitive set N with ${}^{\alpha}N \subseteq N$, and an elementary embedding $j : V_\alpha \to N$. $j \restriction V_\alpha^L$ is an elementary embedding from V_α^L to $N \cap L (\in L)$, and $N \cap L$ is closed under α-sequences in L.

We have $j(\kappa) = j(\mu^+) = (\mu^+)^{V^{\mathbb{P}}}$, and one can check that $j(\kappa)$ is inaccessible in L. Hence $V_{j(\kappa)}^L = L_{j(\kappa)} \subseteq N \cap L$. By Gitman-Schindler's theorem, forcing with $\mathrm{Col}(|V_\alpha^L|)$ over L adds such an elementary embedding.

If κ is generically setwise supercompact, then κ is virtually extendible in L.

We see that κ is virtually jointly setwise supercompact and superstrong in L. If $\kappa > \omega_2$, then $0^\#$ exists and it is O.K. If $\kappa \leq \omega_2$, then κ is successor, say $\kappa = \mu^+$.

Fix $\alpha > \kappa$, and take a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there are a transitive set N with ${}^\alpha N \subseteq N$, and an elementary embedding $j : V_\alpha \to N$. $j \restriction V_\alpha^L$ is an elementary embedding from V_α^L to $N \cap L (\in L)$, and $N \cap L$ is closed under α-sequences in L.

We have $j(\kappa) = j(\mu^+) = (\mu^+)^{V^{\mathbb{P}}}$, and one can check that $j(\kappa)$ is inaccessible in L. Hence $V_{j(\kappa)}^L = L_{j(\kappa)} \subseteq N \cap L$. By Gitman-Schindler's theorem, forcing with $\mathrm{Col}(|V_\alpha^L|)$ over L adds such an elementary embedding.

If κ is generically setwise supercompact, then κ is virtually extendible in L.

We see that κ is virtually jointly setwise supercompact and superstrong in L. If $\kappa > \omega_2$, then $0^{\#}$ exists and it is O.K. If $\kappa \leq \omega_2$, then κ is successor, say $\kappa = \mu^+$.

Fix $\alpha > \kappa$, and take a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there are a transitive set N with ${}^{\alpha}N \subseteq N$, and an elementary embedding $j : V_\alpha \to N$. $j \restriction V_\alpha^L$ is an elementary embedding from V_α^L to $N \cap L (\in L)$, and $N \cap L$ is closed under α-sequences in L.

We have $j(\kappa) = j(\mu^+) = (\mu^+)^{V^{\mathbb{P}}}$, and one can check that $j(\kappa)$ is inaccessible in L. Hence $V_{j(\kappa)}^L = L_{j(\kappa)} \subseteq N \cap L$. By Gitman-Schindler's theorem, forcing with $\mathrm{Col}(|V_\alpha^L|)$ over L adds such an elementary embedding.

Virtually setwise supercompact

Let us consider the virtual version of setwise supercompactness.

Definition

A cardinal κ is *virtually setwise supercompact* if for every regular $\lambda > \kappa$, there are a transitive set N and a poset $\mathbb{P}$ such that

1. ${}^{\lambda}N \subseteq N$.
2. In $V^{\mathbb{P}}$, there is an elementary embedding $j : H_{\lambda}^{V} \to N$ with critical point κ and $\lambda < j(\kappa)$.

Remark

If we require the condition $j``\lambda \in N$ for N, then it is equivalent to the usual supercompact cardinal.

Magidor's characterization

.

Theorem (Magidor)

κ is supercompact if and only if for every $\alpha > \kappa$, there is $\beta < \kappa$ and an elementary embedding $j : V_\beta \to V_\alpha$ with $j(\mathrm{crit}(j)) = \kappa$.

Magidor's characterization lead us to the following virtual large cardinal.

Definition

A cardinal κ is *virtually M-supercompact* if for every $\alpha > \kappa$, there are $\beta < \kappa$ and a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there is an elementary embedding $j : V_\beta \to V_\alpha$ with $j(\mathrm{crit}(j)) = \kappa$.

Theorem (Gitman-Schindler)

κ is virtually setwise supercompact if and only if κ is virtually M-supercompact.

Magidor's characterization

.

Theorem (Magidor)

κ is supercompact if and only if for every $\alpha > \kappa$, there is $\beta < \kappa$ and an elementary embedding $j : V_\beta \to V_\alpha$ with $j(\mathrm{crit}(j)) = \kappa$.

Magidor's characterization lead us to the following virtual large cardinal.

Definition

A cardinal κ is *virtually M-supercompact* if for every $\alpha > \kappa$, there are $\beta < \kappa$ and a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there is an elementary embedding $j : V_\beta \to V_\alpha$ with $j(\mathrm{crit}(j)) = \kappa$.

Theorem (Gitman-Schindler)

κ is virtually setwise supercompact if and only if κ is virtually M-supercompact.

Magidor's characterization

.

Theorem (Magidor)

κ is supercompact if and only if for every $\alpha > \kappa$, there is $\beta < \kappa$ and an elementary embedding $j : V_\beta \to V_\alpha$ with $j(\mathrm{crit}(j)) = \kappa$.

Magidor's characterization lead us to the following virtual large cardinal.

Definition

A cardinal κ is *virtually M-supercompact* if for every $\alpha > \kappa$, there are $\beta < \kappa$ and a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there is an elementary embedding $j : V_\beta \to V_\alpha$ with $j(\mathrm{crit}(j)) = \kappa$.

Theorem (Gitman-Schindler)

κ is virtually setwise supercompact if and only if κ is virtually M-supercompact.

Setwise strong, setwise measurable

Definition

κ is *virtually setwise strong* if for every $\alpha > \kappa$, there are a transitive set N and a poset $\mathbb{P}$ such that:

1. $V_\alpha \subseteq N$.
2. In $V^{\mathbb{P}}$, there is an elementary embedding $j : V_\alpha \to N$ with critical point κ and $\alpha < j(\kappa)$.

Theorem (Gitman-Schindler)

κ is virtually setwise supercompact if and only if κ is virtually setwise strong.

Theorem (Nielsen)

CON($\exists$ virtually setwise supercompact) $\iff$ CON($\exists$ virtually setwise measurable)

Setwise strong, setwise measurable

Definition

κ is *virtually setwise strong* if for every $\alpha > \kappa$, there are a transitive set N and a poset $\mathbb{P}$ such that:

1. $V_\alpha \subseteq N$.
2. In $V^{\mathbb{P}}$, there is an elementary embedding $j : V_\alpha \to N$ with critical point κ and $\alpha < j(\kappa)$.

Theorem (Gitman-Schindler)

κ is virtually setwise supercompact if and only if κ is virtually setwise strong.

Theorem (Nielsen)

CON($\exists$ virtually setwise supercompact) $\iff$ CON($\exists$ virtually setwise measurable)

Setwise strong, setwise measurable

Definition

κ is *virtually setwise strong* if for every $\alpha > \kappa$, there are a transitive set N and a poset $\mathbb{P}$ such that:

1. $V_\alpha \subseteq N$.
2. In $V^{\mathbb{P}}$, there is an elementary embedding $j : V_\alpha \to N$ with critical point κ and $\alpha < j(\kappa)$.

Theorem (Gitman-Schindler)

κ is virtually setwise supercompact if and only if κ is virtually setwise strong.

Theorem (Nielsen)

CON($\exists$ virtually setwise supercompact) $\iff$ CON($\exists$ virtually setwise measurable)

Generic supercompact v.s. Virtual supercompact

Lemma

If κ is generically setwise supercompact, then κ is virtually setwise supercompact in L.

Lemma

If κ is virtually extendible, then V_κ is a model of ZFC+"there are proper class many virtually setwise supercompact cardinals".

Corollary

If κ is a generically setwise supercompact, then L_κ is a model of ZFC+"there are proper class many virtually setwise supercompact cardinals".

Hence CON($\exists$ generically setwise supercompact) is much stronger than CON($\exists$ virtually setwise supercompact).

Generic supercompact v.s. Virtual supercompact

Lemma

If κ is generically setwise supercompact, then κ is virtually setwise supercompact in L.

Lemma

If κ is virtually extendible, then V_κ is a model of ZFC+ "there are proper class many virtually setwise supercompact cardinals".

Corollary

If κ is a generically setwise supercompact, then L_κ is a model of ZFC+ "there are proper class many virtually setwise supercompact cardinals".

Hence CON($\exists$ generically setwise supercompact) is much stronger than CON($\exists$ virtually setwise supercompact).

Generic supercompact v.s. Virtual supercompact

Lemma

If κ is generically setwise supercompact, then κ is virtually setwise supercompact in L.

Lemma

If κ is virtually extendible, then V_κ is a model of ZFC+ "there are proper class many virtually setwise supercompact cardinals".

Corollary

If κ is a generically setwise supercompact, then L_κ is a model of ZFC+ "there are proper class many virtually setwise supercompact cardinals".
Hence CON($\exists$ generically setwise supercompact) is much stronger than CON($\exists$ virtually setwise supercompact).

Setwise tall

Definition (Hamkins)

A cardinal κ is *tall* if for every $\alpha > \kappa$, there are a transitive class N with ${}^{\kappa}N \subseteq N$ and an elementary embedding $j : V \to N$ with critical point κ and $\alpha < j(\kappa)$.

Definition

- κ is *virtually setwise tall* if for every regular $\lambda > \kappa$, there are a transitive set N with ${}^{\kappa}N \subseteq N$ and a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there is an elementary embedding $j : H_{\lambda} \to N$ with critical point κ and $\lambda < j(\kappa)$.
- κ is *generically setwise tall* if for every regular $\lambda > \kappa$, there is a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there are a transitive set N with ${}^{\kappa}N \subseteq N$ and an elementary embedding $j : H_{\lambda}^{V} \to N$ with $\lambda < j(\kappa)$.

Setwise tall

Definition (Hamkins)

A cardinal κ is *tall* if for every $\alpha > \kappa$, there are a transitive class N with ${}^{\kappa}N \subseteq N$ and an elementary embedding $j : V \to N$ with critical point κ and $\alpha < j(\kappa)$.

Definition

- κ is *virtually setwise tall* if for every regular $\lambda > \kappa$, there are a transitive set N with ${}^{\kappa}N \subseteq N$ and a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there is an elementary embedding $j : H_{\lambda} \to N$ with critical point κ and $\lambda < j(\kappa)$.
- κ is *generically setwise tall* if for every regular $\lambda > \kappa$, there is a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there are a transitive set N with ${}^{\kappa}N \subseteq N$ and an elementary embedding $j : H_{\lambda}^{V} \to N$ with $\lambda < j(\kappa)$.

- If there is a generically setwise tall $> \omega_2$, then $0^\#$ exists.
- If κ is successor, then κ is generically setwise tall $\iff$ generically setwise supercompact.

Corollary

CON($\exists$ generically setwise supercompact)
$\iff$ CON($\exists$ generically setwise tall).

- If there is a generically setwise tall $> \omega_2$, then $0^{\#}$ exists.
- If κ is successor, then κ is generically setwise tall $\iff$ generically setwise supercompact.

Corollary

CON($\exists$ generically setwise supercompact)
$\iff$ CON($\exists$ generically setwise tall).

Strong v.s. Tall

- Tall is not equivalent to strong in general.

Theorem (Hamkins)

CON($\exists$ tall) $\iff$ CON($\exists$ strong)

Proposition

1. Every virtually setwise strong cardinal is virtually setwise tall.
2. If κ is virtually setwise tall, then so is in L.
3. In L, every virtually setwise tall is virtually setwise strong.
4. CON($\exists$ virtually setwise strong) $\iff$ CON($\exists$ virtually setwise tall)

Question

Is virtually setwise tall always equivalent to virtually setwise strong?

Strong v.s. Tall

- Tall is not equivalent to strong in general.

Theorem (Hamkins)

CON($\exists$ tall) $\iff$ CON($\exists$ strong)

Proposition

1. Every virtually setwise strong cardinal is virtually setwise tall.
2. If κ is virtually setwise tall, then so is in L.
3. In L, every virtually setwise tall is virtually setwise strong.
4. CON($\exists$ virtually setwise strong) $\iff$ CON($\exists$ virtually setwise tall)

Question

Is virtually setwise tall always equivalent to virtually setwise strong?

Strong v.s. Tall

- Tall is not equivalent to strong in general.

Theorem (Hamkins)

CON($\exists$ tall) $\iff$ CON($\exists$ strong)

Proposition

1. Every virtually setwise strong cardinal is virtually setwise tall.
2. If κ is virtually setwise tall, then so is in L.
3. In L, every virtually setwise tall is virtually setwise strong.
4. CON($\exists$ virtually setwise strong) $\iff$ CON($\exists$ virtually setwise tall)

Question

Is virtually setwise tall always equivalent to virtually setwise strong?

Generically setwise strong

Definition

κ is *generically setwise strong* if for every $\alpha \geq \kappa$, there is a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there are a transitive set N with $V_\alpha^{\mathbb{P}} \subseteq N$ and an elementary embedding $j : V_\alpha \to N$ with critical point κ and $\alpha < j(\kappa)$.

- If there is a generically setwise strong $> \omega_2$, then $0^\#$ exists.
- If κ is generically setwise strong, the κ is virtually jointly setwise strong and superstrong in L.

Corollary

The following are equiconsistent:

1. $\exists$ generically setwise tall.
2. $\exists$ generically setwise strong.
3. $\exists$ generically setwise supercompact.

Generically setwise strong

Definition

κ is *generically setwise strong* if for every $\alpha \geq \kappa$, there is a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there are a transitive set N with $V_{\alpha}^{\mathbb{P}} \subseteq N$ and an elementary embedding $j : V_{\alpha} \to N$ with critical point κ and $\alpha < j(\kappa)$.

- If there is a generically setwise strong $> \omega_2$, then $0^{\#}$ exists.
- If κ is generically setwise strong, the κ is virtually jointly setwise strong and superstrong in L.

Corollary

The following are equiconsistent:

1. $\exists$ generically setwise tall.
2. $\exists$ generically setwise strong.
3. $\exists$ generically setwise supercompact.

Generically setwise strong

Definition

κ is *generically setwise strong* if for every $\alpha \geq \kappa$, there is a poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there are a transitive set N with $V_\alpha^{\mathbb{P}} \subseteq N$ and an elementary embedding $j : V_\alpha \to N$ with critical point κ and $\alpha < j(\kappa)$.

- If there is a generically setwise strong $> \omega_2$, then $0^\#$ exists.
- If κ is generically setwise strong, the κ is virtually jointly setwise strong and superstrong in L.

Corollary

The following are equiconsistent:

1. $\exists$ generically setwise tall.
2. $\exists$ generically setwise strong.
3. $\exists$ generically setwise supercompact.

Conclusion

Generically setwise supercompact $> \omega_2$

$\Downarrow \not\Uparrow$

$0^\#$

$\Downarrow \not\Uparrow$

Generically extendible (for ω_1)
Generically setwise supercompact (for ω_1)
Generically setwise tall (for ω_1)
Generically setwise strong (for ω_1)

Virtually extendible

$\Downarrow \not\Uparrow$

Virtually setwise supercompact
Virtually setwise tall
Virtually setwise strong

References

- J. Bagaria, J. D. Hamkins, K. Tsaprounis, T. Usuba, "Superstrong and other large cardinals are never Laver indestructible", Arch. Math. Logic. 55 (1–2), pp. 19–35, 2016.
- V. Gitman, R. Schindler, "Virtual large cardinals", Ann. Pure and Appl. Logic Vol. 169, Issue 12, 2018.
- J. D. Hamkins, "Tall cardinals", Math. Logic Q., vol. 55, iss. 1, 2009.
- D. Ikegami, J. Väänänen, "Boolean-Valued Second-Order Logic", Notre Dame J. Form. Logic 56(1), pp.167–190, 2015.
- R. Laver, "Making the supercompactness of κ indestructible under κ-directed closed forcing", Isr. J. Math. Volume 29, pp.385–388, 1978.
- D. S. Nielsen, "Virtual set theory", Ph.D. Thesis.
- P. Schlicht, "A variant of generic supercompactness", Talk in RIMS set theory workshop 2019. Slide is available at `http://www.sic.shibaura-it.ac.jp/~ikegami/Schlicht.pdf`
- K. Tsaprounis, "Elementary chains and $C(n)$-cardinals", Arch. Math. Logic 53(1–2), pp.89–118, 2014