Preserving failures of simple fragments of Martin's axiom

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Definition

Let $\mathcal{C}$ be a class of posets, κ an infinite cardinal.

$\mathrm{MA}_\kappa(\mathcal{C})$ For any $\mathbb{P} \in \mathcal{C}$ and $\mathcal{D} \in [\mathrm{dense}(\mathbb{P})]^{\leq\kappa}$, there is some $(\mathbb{P}, \mathcal{D})$-generic filter.

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$\mathrm{MA}_{\aleph_0}(\mathrm{all})$ holds, but $\mathrm{MA}_{\mathfrak{c}}(\mathrm{Cohen})$ fails.

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2. $\mathfrak{m}(\mathrm{all}) = \aleph_1$, so $\aleph_1 \leq \mathfrak{m}(\mathcal{C})$.
3. $\mathfrak{m}(\mathrm{countable}) = \mathfrak{m}(\mathrm{Cohen}) = \mathrm{cov}(\mathcal{M})$.

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Goal

Preserve failures $(\mathbb{P}, \mathcal{D})$ of $\mathrm{MA}(\mathcal{C})$ in forcing extensions,
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A Suslin tree $\mathbb{T}$ is a failure of $\mathrm{MA}(\mathrm{ccc})$
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$\rightarrow$ preserve $\mathfrak{m}(\mathrm{ccc}) = \aleph_1$.

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We look at properties Γ *of subsets of posets,*
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$$\forall B \in [\mathbb{P}]^{\mu} \exists A \in [B]^{\mu} (A \in \Gamma(\mathbb{P})).$$

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However, not true in general: Γ_0 is a counter-example.

Bonus track

Let μ and infinite cardinal and $\theta > \aleph_0$ regular.

M. 2019

Properties on Γ can be determined so that:

1. Any FS (finite support) product of θ-Γ-Knaster posets is θ-Γ-Knaster.

Matrix iterations with vertical support restrictions. (Section 5. "Bonus track: linkedness properties")

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Λ_n and $\Lambda_{<\omega}$ satisfy both, but Γ_0 satisfies only (2).

Very standard classes

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Notation

1. n-Knaster means $(\aleph_1, n)$-Knaster for $2 \leq n < \omega$.
2. 1-Knaster means ccc.
3. ω-Knaster means precaliber $\aleph_1$.
4. $\mathfrak{m}_\gamma := \mathfrak{m}(\gamma\text{-Kn.})$ $(1 \leq \gamma \leq \omega)$.

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Diagrams

$$\Gamma_0 \longleftarrow \Lambda_2 \longleftarrow \Lambda_3 \longleftarrow \cdots \longleftarrow \Lambda_{<\omega}$$

$$\begin{array}{ccccccccc}
 & & \sigma\text{-}2\text{-linked} & \longleftarrow & \sigma\text{-}3\text{-linked} & \longleftarrow & \cdots & \longleftarrow & \sigma\text{-centered} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{ccc} & \longleftarrow & 2\text{-Knaster} & \longleftarrow & 3\text{-Knaster} & \longleftarrow & \cdots & \longleftarrow & \text{precaliber } \aleph_1
\end{array}$$

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 & & \mathfrak{m}(\sigma\text{-}2\text{-lk.}) & \longrightarrow & \mathfrak{m}(\sigma\text{-}3\text{-lk.}) & \longrightarrow & \cdots & \longrightarrow & \mathfrak{p} = \mathfrak{t} = \mathfrak{m}(\sigma\text{-ct.}) \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
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Fact (ess. Kunen, Rowbottom, Solovay)

Assume that $\mathcal{C} \subseteq \mathrm{ccc}$ is closed under countable FS-products and under *cones* (if $\mathbb{P} \in \mathcal{C}$ and $p \in \mathbb{P}$ then $\{q \in \mathbb{P} : q \leq p\} \in \mathcal{C}$).

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Corollary

$|\{\mathfrak{m}_\gamma : 1 \leq \gamma \leq \omega\}| \leq 2$

Playground

Theorem (Goldstern & Kellner & M. & Shelah 2021)

For any $1 \leq k \leq \omega$ and $\lambda \geq \aleph_1$ regular there is a FS iteration forcing $\mathfrak{m}_\ell = \aleph_1$ for $\ell < k$, and $\mathfrak{m}_k = \mathfrak{m}_\omega = \lambda$.
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We first add many Cohen reals (the value we want for $\mathfrak{c}$), followed by posets that are σ-n-linked for all $n < \omega$ (for Cichoń's maximum, and $\mathfrak{p}$), plust other stuff depending on each case.

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Note: $\mathfrak{m}(\sigma\text{-}n\text{-lk.}) \leq \mathrm{add}(\mathcal{N})$ for all $2 \leq n < \omega$.

Forcing $\mathfrak{m}_\omega = \aleph_1$

Case $\lambda = \aleph_1$: No more is needed.

Devlin & Shelah 1978

$\mathfrak{m}_\omega > \aleph_1$ implies that any ladder system coloring can be uniformalized.
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Barnett 1992

After one Cohen real, there is a ladder system coloring that cannot be uniformalized even in further σ-linked forcing extensions.

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What about $\mathfrak{m}_{k-1} = \aleph_1$?: the iteration is $(\aleph_1, k)$-Knaster, so the same would ensure $\mathfrak{m}_{k-1} \leq \aleph_1$.

Quite strong nwd sets

Assume $n \geq 3$.

Definition (Todorčević 1980's)

1. $X \subseteq n^\omega$ is $<n$-ary, nwd^+ in this talk, if

$$\forall s \in n^{<\omega} \exists i < n([s^\frown\langle i\rangle] \cap X = \emptyset)$$

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3. For $Z \subseteq n^\omega$ define the poset $\mathbb{Q}_Z$ whose conditions are functions $p : u \to \omega$ with $u \subseteq Z$ finite s.t. $p^{-1}(n)$ is nwd^+ for all $n < \omega$. The order is $\supseteq$.

Preserving Luzin$^+$ sets

Assume $n \geq 3$, $\lambda \geq \aleph_1$ regular.

Lemma (Todorčević 1980's)

If $Z \in [n^\omega]^\lambda$ is λ-Luzin$^+$ then $(\mathbb{Q}_Z, \mathcal{D})$ fails $\mathrm{MA}((n-1)$-Kn.$)$ where $\mathcal{D} := \{D_z : z \in Z\}$, $D_z := \{q \in \mathbb{Q}_Z : z \in \mathrm{dom} q\}$.

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Case $k = 1$: Since $\aleph_1 < \lambda \leq \mathfrak{m}_1$ we have $\mathfrak{m}_1 = \mathfrak{m}_\omega$, so it is enough to get $\mathfrak{m}_2 \leq \lambda$.
This follows from Todorčević's theorem applied to $n = 3$.

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Challenge

Preserve a failure of precaliber $\aleph_1$ of size λ in $(\lambda, 2)$-Knaster extensions.

Colorings with many types

Definition (GKMS 2021)

$(c, \bar{d})$ is a λ-type coloring if it satisfies

1. $c : [\lambda]^2 \to \{0,1\}$, $\bar{d} = \langle d_\alpha : \alpha < \lambda \rangle$, $d_\alpha : [\alpha]^{<\aleph_0} \to \omega$.

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4. For any 1-homogeneous (under c) $s, t \in [\alpha]^{<\aleph_0}$, if $d_\alpha(s) = d_\alpha(t)$ (same α-type) then $s \cup t$ is 1-homogeneous.

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(4) $\Rightarrow$ $\mathbb{Q}_{(c,\bar{d})}$ has precaliber $\aleph_1$.

Even precaliber μ for any regular $\aleph_1 \leq \mu < \mathrm{cf}(\lambda) = \lambda$.

Main Lemma

Lemma (GKMS 2021)

If $(c, \bar{d})$ is a λ-type coloring then $(\mathbb{Q}_{(c,d)}, \mathcal{D})$ is a failure of $\mathrm{MA}(\text{prec.})$ where $\mathcal{D} = \{D_\alpha : \alpha < \lambda\}$, $D_\alpha := \{u \in \mathbb{Q}_{(c,\bar{d})} : u \nsubseteq \alpha\}$.

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Assume $\mathbb{P}$ is $(\lambda, 2)$-Knaster, $p \in \mathbb{P}$, $p \Vdash$ "$\dot{A} = \{\dot{a}_\alpha : \alpha < \lambda\}$ is as in (3)".

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So any $q \leq p_\alpha, p_\beta$ forces that $\dot{a}_\alpha = b_\alpha$ and $\dot{a}_\beta = b_\beta$ are as desired.

Where is the coloring?

Theorem (GKMS 2021)

For any regular $\lambda > \aleph_1$ there is some precaliber $\aleph_1$ poset $\mathbb{P}^*_\lambda$ of size λ adding a λ-type coloring.
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So $\mathbb{P}^*_\lambda$ should be included at the beginning of the iteration.

About $\mathfrak{p}$ and 'the rest'

At this point our iteration separates

1. the $\mathfrak{m}_k$ numbers (in two sections),
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After intersecting with models **[GKMS 2020]** we obtain Cichoń's maximum while preserving (1) and forcing $\mathfrak{m}_\omega < \mathfrak{p} = \mathfrak{h} = \mathfrak{g} < \mathrm{add}(\mathcal{N})$.

Separating $\mathfrak{p}$ and $\mathfrak{h}$

Lemma (ess. Dow & Shelah)

Let $\aleph_1 \leq \mu = \mu^{<\mu} \leq \kappa$ be uncountable regular, $\mathbb{P}$ μ-cc forcing $\mu \leq \mathfrak{p}$.

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3. $\mathbb{P} * ({}^{<\mu}\mu)^V$ preserves the values of $\mathfrak{m}_k$ forced by $\mathbb{P}$, as long as they are $\leq \mu$.

Many more stuff