# Preserving failures of simple fragments of Martin's axiom

Diego A. Mejía diego.mejia@shizuoka.ac.jp

Shizuoka University

Joint work with Martin Goldstern, Jakob Kellner, and Saharon Shelah

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 $MA_{\aleph_0}(all)$  holds, but  $MA_{\mathfrak{c}}(Cohen)$  fails.

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Preserve failures  $(\mathbb{P}, \mathcal{D})$  of  $\mathrm{MA}(\mathcal{C})$  in forcing extensions, i.e. preservation of  $\mathfrak{m}(\mathcal{C}) \leq \kappa$ .

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A Suslin tree  $\mathbb T$  is a failure of  $\mathrm{MA}(\mathrm{ccc})$ 

(with 
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- **2** A poset  $\mathbb{P}$  is  $\mu$ - $\Gamma$ -Knaster if

$$\forall B \in [\mathbb{P}]^{\mu} \exists A \in [B]^{\mu} (A \in \Gamma(\mathbb{P})).$$

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However, not true in general:  $\Gamma_0$  is a counter-example.



Let  $\mu$  and infinite cardinal and  $\theta > \aleph_0$  regular.

#### M. 2019

Properties on  $\Gamma$  can be determined so that:

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 $\Lambda_n$  and  $\Lambda_{<\omega}$  satisfy both, but  $\Gamma_0$  satisfies only (2).

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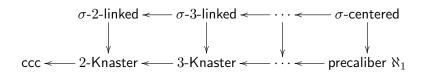
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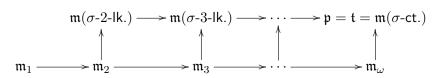
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## **Diagrams**

$$\Gamma_0 \longleftarrow \Lambda_2 \longleftarrow \Lambda_3 \longleftarrow \cdots \longleftarrow \Lambda_{<\omega}$$





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## Fact (ess. Kunen, Rowbottom, Solovay)

Assume that  $\mathcal{C} \subseteq \csc$  is closed under countable FS-products and under cones (if  $\mathbb{P} \in \mathcal{C}$  and  $p \in \mathbb{P}$  then  $\{q \in \mathbb{P} : q \leq p\} \in \mathcal{C}$ ).

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### Corollary

$$|\{\mathfrak{m}_{\gamma}: 1 \leq \gamma \leq \omega\}| \leq 2$$

## Theorem (Goldstern & Kellner & M. & Shelah 2021)

For any  $1 \leq k \leq \omega$  and  $\lambda \geq \aleph_1$  regular there is a FS iteration forcing  $\mathfrak{m}_\ell = \aleph_1$  for  $\ell < k$ , and  $\mathfrak{m}_k = \mathfrak{m}_\omega = \lambda$ . In addition we can force  $\lambda < \mathfrak{p} < \operatorname{add}(\mathcal{N})$ , along with Cichoń's maximum.

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We first add many Cohen reals (the value we want for  $\mathfrak c$ ), followed by posets that are  $\sigma$ -n-linked for all  $n<\omega$  (for Cichoń's maximum, and  $\mathfrak p$ ), plust other stuff depending on each case.

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Note:  $\mathfrak{m}(\sigma$ -n-lk.)  $\leq \operatorname{add}(\mathcal{N})$  for all  $2 \leq n < \omega$ .

# Forcing $\mathfrak{m}_{\omega} = \aleph_1$

Case  $\lambda = \aleph_1$ : No more is needed.

### Devlin & Shelah 1978

 $\mathfrak{m}_{\omega} > \aleph_1$  implies that any ladder system coloring can be uniformalized. I.e. Any ladder system coloring that cannot be uniformalized is a failure of  $\mathrm{MA}(\mathsf{prec.})$ .

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#### Barnett 1992

After one Cohen real, there is a ladder system coloring that cannot be uniformalized even in further  $\sigma$ -linked forcing extensions.

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What about  $\mathfrak{m}_{k-1} = \aleph_1$ ?: the iteration is  $(\aleph_1, k)$ -Knaster, so the same would ensure  $\mathfrak{m}_{k-1} \leq \aleph_1$ .

# Quite strong nwd sets

#### Assume $n \ge 3$ .

## Definition (Todorčević 1980's)

**1**  $X \subseteq n^{\omega}$  is < n-ary,  $\underline{\text{nwd}}^+$  in this talk, if

$$\forall s \in n^{<\omega} \exists i < n([s^{\hat{}}\langle i \rangle] \cap X = \emptyset)$$

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- For Z ⊆ n<sup>ω</sup> define the poset Q<sub>Z</sub> whose conditions are functions  $p: u \to ω$  with  $u \subseteq Z$  finite s.t.  $p^{-1}(n)$  is  $nwd^+$  for all n < ω. The order is ⊇.

Assume  $n \geq 3$ ,  $\lambda \geq \aleph_1$  regular.

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If  $Z \in [n^{\omega}]^{\lambda}$  is  $\lambda$ -Luzin<sup>+</sup> then  $(\mathbb{Q}_Z, \mathcal{D})$  fails  $\operatorname{MA}((n-1)$ -Kn.) where  $\mathcal{D} := \{D_z : z \in Z\}, \ D_z := \{q \in \mathbb{Q}_Z : z \in \operatorname{dom} q\}.$ 

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#### Lower cases

So the following remains to be checked:

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- [Shelah '84, Velleman '84, Todorčević 89] There is a Suslin tree after one Cohen real.

<u>Case k=1</u>: Since  $\aleph_1 < \lambda \leq \mathfrak{m}_1$  we have  $\mathfrak{m}_1 = \mathfrak{m}_{\omega}$ , so it is enough to get  $\mathfrak{m}_2 \leq \lambda$ .

This follows from Todorčević's theorem applied to n=3.

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#### Challenge

Preserve a failure of precaliber  $\aleph_1$  of size  $\lambda$  in  $(\lambda, 2)$ -Knaster extensions.

### Definition (GKMS 2021)

 $(c, \bar{d})$  is a  $\lambda$ -type coloring if it satisfies

Define  $\mathbb{Q}_{(c,\bar{d})}:=\{u\in[\lambda]^{<\aleph_0}: \text{u is 1-homogeneous}\}$ , ordered by  $\supseteq$ .

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- $(3) \Rightarrow$  This homogeneous set is not in the ground model.

### Definition (GKMS 2021)

 $(c, \bar{d})$  is a  $\lambda$ -type coloring if it satisfies

- $\textbf{ 2 For any } u \in [\lambda]^{<\aleph_0} \text{, } |\{\eta < \lambda: \, \forall \xi \in u(c(\{\xi,\eta\})=1)\}| = \lambda.$
- $\textbf{ If } A \subseteq [\lambda]^{<\aleph_0} \text{ is a family of pairwise disjoint sets and } |A| = \lambda \text{ then } \\ \exists u \neq v \text{ in } A \text{ s.t. } \forall \xi \in u \forall \eta \in v(c(\{\xi,\eta\}) = 0).$
- $\bullet \ \, \text{For any 1-homogeneous (under $c$) $s,t\in[\alpha]^{<\aleph_0}$, if $d_\alpha(s)=d_\alpha(t)$ (same $\alpha$-type) then $s\cup t$ is 1-homogeneous.}$

Define  $\mathbb{Q}_{(c,\bar{d})}:=\{u\in[\lambda]^{<\aleph_0}: \text{u is 1-homogeneous}\}$ , ordered by  $\supseteq$ .

- (2)  $\Rightarrow \mathbb{Q}_{(c,\bar{d})}$  adds a 1-homogeneous set of size  $\lambda$ .
- $(3) \Rightarrow$  This homogeneous set is not in the ground model.
- $(4) \Rightarrow \mathbb{Q}_{(c,\bar{d})}$  has precaliber  $\aleph_1$ .

Even precaliber  $\mu$  for any regular  $\aleph_1 \leq \mu < \mathrm{cf}(\lambda) = \lambda$ .

#### Main Lemma

#### Lemma (GKMS 2021)

If  $(c, \bar{d})$  is a  $\lambda$ -type coloring then  $(\mathbb{Q}_{(c,d)}, \mathcal{D})$  is a failure of  $\mathrm{MA}(\mathsf{prec.})$  where  $\mathcal{D} = \{D_\alpha : \ \alpha < \lambda\}, \ D_\alpha := \{u \in \mathbb{Q}_{(c,\bar{d})} : \ u \nsubseteq \alpha\}.$ 

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s.t.  $\forall \xi \in b_{\alpha} \forall \eta \in b_{\beta}(c(\{\xi, \eta\}) = 0).$ 

So any  $q \leq p_{\alpha}, p_{\beta}$  forces that  $\dot{a}_{\alpha} = b_{\alpha}$  and  $\dot{a}_{\beta} = b_{\beta}$  are as desired.

# Where is the coloring?

### Theorem (GKMS 2021)

For any regular  $\lambda > \aleph_1$  there is some precaliber  $\aleph_1$  poset  $\mathbb{P}^*_{\lambda}$  of size  $\lambda$  adding a  $\lambda$ -type coloring.

Even more,  $\mathbb{P}^*_{\lambda}$  has precaliber  $\mu$  for any regular  $\mu \geq \aleph_1$ .

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So  $\mathbb{P}^*_{\lambda}$  should be included at the beginning of the iteration.

## About p and 'the rest'

At this point our iteration separates

- the  $\mathfrak{m}_k$  numbers (in two sections),
- the left side of Cichoń's diagram
- $oldsymbol{0}$  and forces  $\mathfrak{p}=\mathfrak{b}$ .

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After intersecting with models **[GKMS 2020]** we obtain Cichoń's maximum while preserving (1) and forcing  $\mathfrak{m}_{\omega} < \mathfrak{p} = \mathfrak{h} = \mathfrak{g} < \operatorname{add}(\mathcal{N})$ .

## Separating $\mathfrak{p}$ and $\mathfrak{h}$

### Lemma (ess. Dow & Shelah)

Let  $\aleph_1 \leq \mu = \mu^{<\mu} \leq \kappa$  be uncountable regular,  $\mathbb{P}$   $\mu$ -cc forcing  $\mu \leq \mathfrak{p}$ .

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In the previous lemma:

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# Many more stuff

