Scott Processes Revisited

Paul Larson

Department of Mathematics
Miami University
Oxford, Ohio 45056

larsonpb@miamioh.edu

March 10, 2021

Relational Vocabularies

Througout this talk, τ will stand for a countable relational vocabulary (which includes $=$ and a 0-ary relation symbol $\top$).

We use the symbols x_i ($i \in \omega$) as variables, and let X_n denote

$$\{x_0, \ldots, x_{n-1}\}.$$

Infinitary Languages

Let κ be an infinite cardinal. The formulas in $\mathcal{L}_{\kappa,\aleph_0}(\tau)$ are built up from the atomic formulas using

- negation ($\neg$)
- quantifiers ($\exists x_i$ and $\forall x_i$, for $i \in \omega$), and
- unordered conjunctions and disjunctions of cardinality less than κ (when the resulting formula has finitely many free variables).

$\mathcal{L}_{\infty,\aleph_0}(\tau)$ is the class of all formulas in any $\mathcal{L}_{\kappa,\aleph_0}(\tau)$.

The Scott analysis of a τ-structure

Let M be a τ-structure.

For each ordinal α and each finite tuple $\bar{a}$ of distinct elements of M we define the $|\bar{a}|$-ary formula

$$\phi^{M}_{\bar{a},\alpha} \in \mathcal{L}_{|M|^+,\aleph_0}(\tau)$$

as follows.

- $\phi^M_{\bar{a},0}$ is the conjunction of all atomic and negated atomic formulas satisfied by $\bar{a}$ in M, using the free variables

$$x_0, \ldots, x_{|a|-1};$$

- $\phi^M_{\bar{a},\alpha+1}$ is the conjunction of the following three formulas:
  - $\phi^M_{\bar{a},\alpha}$,
  - $\bigwedge_{c \in M \setminus \bar{a}} \exists x_{|\bar{a}|} \phi^M_{\bar{a}^\frown \langle c \rangle, \alpha}$,
  - $\forall x_{|a|} \notin \{x_0, \ldots, x_{|a|-1}\} \bigvee_{c \in M \setminus \bar{a}} \phi^M_{\bar{a}^\frown \langle c \rangle, \alpha}$;
- for limit ordinals β, $\phi^M_{\bar{a},\beta} = \bigwedge_{\alpha < \beta} \phi^M_{\bar{a},\alpha}$.

$E(\phi^M_{\bar{a},\alpha+1})$

We let $E(\phi^M_{\bar{a},\alpha+1})$ denote the set

$$\{\phi^M_{\bar{a}^\frown\langle c\rangle,\alpha} : c \in M \setminus \bar{a}\}.$$

The formula $\phi^M_{\bar{a},\alpha+1}$ is then the conjunction of:

- $\phi^M_{\bar{a},\alpha}$,
- $\bigwedge_{\phi\in E(\phi^M_{\bar{a},\alpha+1})} \exists x_{|\bar{a}|}\psi$,
- $\forall x_{|a|} \notin \{x_0,\ldots,x_{|a|-1}\} \bigvee_{\psi\in E(\phi^M_{\bar{a},\alpha+1})} \psi$;

We call $\phi^M_{\bar{a},\alpha}$ the *Scott formula* of $\bar{a}$ *in* M *at level* α.

For each finite tuple $\bar{a}$ from M, and each ordinal α,

$$M \models \phi^M_{\bar{a},\alpha}(\bar{a}).$$

For all finite tuples $\bar{a}$, $\bar{b}$ from M, and all ordinals $\alpha < \beta$, if

$$\phi^M_{\bar{a},\beta} = \phi^M_{\bar{b},\beta}$$

then

$$\phi^M_{\bar{a},\alpha} = \phi^M_{\bar{b},\alpha}.$$

For each ordinal α, we let $\Phi_\alpha(M)$ denote

$$\{\phi^M_{\bar{a},\alpha} : \bar{a} \in M^{<\omega}\}$$

and we call

$$\langle \Phi_\alpha(M) : \alpha \in \mathrm{Ord} \rangle$$

the *Scott process* of M.

We also write $\mathrm{SP}_\beta(M)$ for $\langle \Phi_\alpha(M) : \alpha < \beta \rangle$.

Scott sentences

For some $\alpha < |M|^+$, for all finite tuples $\bar{a}$, $\bar{b}$ from M, if

$$\phi^M_{\bar{a},\alpha} = \phi^M_{\bar{b},\alpha}$$

then

$$\phi^M_{\bar{a},\alpha+1} = \phi^M_{\bar{b},\alpha+1}.$$

The least such α is called the *Scott rank* of M.

The (canonical) *Scott sentence* of M is the conjunction of $\phi^M_{\langle\rangle,\alpha+1}$ with the conjunction over all finite tuples $\bar{a}$ from M of the sentence

$$\forall x_0, \ldots, x_{|\bar{a}|-1} \, (\phi^M_{\bar{a},\alpha} \rightarrow \phi^M_{\bar{a},\alpha+1})$$

Scott's Isomorphism Theorem

Theorem (Scott)

Countable τ-structures with the same Scott sentence are isomorphic.

Quantifier Depth

The quantifier depth $\mathrm{qd}(\phi)$ of a formula ϕ of $\mathcal{L}_{\infty,\aleph_0}(\tau)$ is defined as follows.

- An atomic formula has quantifier depth 0.
- $\mathrm{qd}(\neg\phi) = \mathrm{qd}(\phi)$.
- $\mathrm{qd}(\bigwedge_{\alpha<\kappa} \phi_\alpha) = \sup\{\mathrm{qd}(\phi_\alpha) : \alpha < \kappa\}$.
- $\mathrm{qd}(\bigvee_{\alpha<\kappa} \phi_\alpha) = \sup\{\mathrm{qd}(\phi_\alpha) : \alpha < \kappa\}$.
- $\mathrm{qd}(\exists x_i\phi) = \mathrm{qd}(\forall x_i\phi) = \mathrm{qd}(\phi) + 1$.

Given τ-structures M and N, $n \in \omega$, an ordinal α and n-tuples $\bar{a}$ from M and $\bar{b}$ from N, each consisting of distinct elements,

$$\phi^M_{\bar{a},\alpha} = \phi^N_{\bar{b},\alpha}$$

if and only if, for each n-ary $\mathcal{L}_{\infty,\aleph_0}(\tau)$ formula ψ of quantifier depth at most α,

$$M \models \psi(\bar{a})$$

if and only if

$$N \models \psi(\bar{b}).$$

It follows that if M is a τ-structure of Scott rank α, then

$$\phi^{M}_{\langle\rangle,\alpha+\omega}$$

characterizes M up to isomorphism.

Structures on ω

The set of τ-structures with domain ω is naturally conceived as a Polish space $\mathcal{S}_\tau$, where a typical subbasic clopen set is the set of such structures satisfying $R(\bar{n})$, for R a relational symbol from τ, and $\bar{n}$ a finite tuple from ω.

By a theorem of Lopez-Escobar a set $\mathcal{S} \subseteq \mathcal{S}_\tau$ is Borel if and only if, for some $\phi \in \mathcal{L}_{\aleph_1,\aleph_0}(\tau)$,

$$\mathcal{S} = \{M \in \mathcal{S} \mid M \models \phi\}.$$

Vaught's Conjecture

Vaught's Conjecture (1961) says that for all $\phi \in \mathcal{L}_{\aleph_1,\aleph_0}(\tau)$, if ϕ has uncountably many nonisomorphic countable models, then ϕ has a perfect set of nonisomorphic countable models.

The conjecture has been verified for various classes of structures, for instance, trees (Steel, 1978).

A counterexample has been claimed by R. Knight (2002), but not (as far as I know) verified by the community.

Vaught's Conjecture for analytic families

A natural generalization of Vaught's Conjecture says that every analytic family of τ-structures on ω has a perfect set of nonisomorphic countable models if it has uncountably many.

This statement is false, however.

Counterexamples

(H. Friedman) The ordinals of countable ω-models of Kripke-Platek set theory are all ordertypes of the form α or $\alpha + \alpha \cdot \eta$, for α a countable admissible ordinal and η the ordertype of the rationals.

(Kunen) The countable 1-transitive linear orders are all ordertypes of the form $\mathbb{Z}^\alpha$ or $\mathbb{Z}^\alpha \cdot \eta$, for α a countable ordinal.

Two Classical Theorems

Theorem (Harrington, 1970's?). Every counterexample to Vaught's Conjecture has models of cofinally many Scott ranks below ω_2.

Theorem (Sacks, 1982). Every counterexample to Vaught's Conjecture has nonisomorphic countable models of the same Scott rank.

Ψ^n_α and V_α

One can recursively define sets

$$\Psi^n_\alpha \subset \mathcal{L}_{\infty,\aleph_0}(\tau)$$

(for $\alpha \in \mathrm{Ord}$, $n \in \omega$) and class-sized functions V_α ($\alpha \in \mathrm{Ord}$) such that whenever M is a τ-structure, $n \in \omega$, $\bar{a}$ is an injective n-tuple from M and $\alpha \leq \beta$ are ordinals, then

$$\phi^M_{\bar{a},\beta} \in \Psi^n_\beta$$

and

$$\phi^M_{\bar{a},\alpha} = V_\alpha(\phi^M_{\bar{a},\beta}).$$

Part 1 of the definition

- Each Ψ_0^n consists of all conjunctions consisting of, for each atomic formula from τ in the variables $x_0, \ldots, x_{n-1}$, either the formula or its negation, including (when $n \geq 2$) all instances of $\neg(x_i = x_j)$ for $i \neq j$.
- Each $\theta \in \Psi_{\alpha+1}^n$ consists of all conjunctions of the three following formulas, for some $\phi \in \Psi_\alpha^n$ and some $E \subseteq \Psi_\alpha^{n+1}$, which we call $E(\theta)$:
  - ϕ;
  - $\bigwedge_{\psi \in E} \exists x_n \psi$;
  - $\forall x_n \notin \{x_0, \ldots, x_{n-1}\} \bigvee_{\psi \in E} \psi$.

Part 2

- For all ordinals α, $\Psi_\alpha = \bigcup_{n\in\omega} \Psi^n_\alpha$.
- $\Psi = \bigcup_{\alpha\in\mathrm{Ord}} \Psi_\alpha$.
- For each $n \in \omega$, $\Psi^n = \bigcup_{\alpha\in\mathrm{Ord}} \Psi^n_\alpha$.
- Each V_α has domain $\bigcup_{\beta\geq\alpha} \Psi_\beta$ and range Ψ_α, and is the identity function on Ψ_α.
- $V_\alpha(\theta) = \phi$ for θ and ϕ as in the definition of $\Psi^n_{\alpha+1}$.
- For all $\alpha \leq \beta$, $V_\alpha = V_\alpha \circ V_\beta$.

Part 3

- For limit ordinals β, each Ψ^n_β consists of all conjunctions of the form
$$\bigwedge_{\alpha<\beta} \phi_\alpha,$$
where each ϕ_α is in Ψ^n_α and $V_\gamma(\phi_\alpha) = \phi_\gamma$ whenever $\gamma \leq \alpha < \beta$.

- For limit ordinals β, for all $\gamma < \beta$,
$$V_\gamma(\bigwedge_{\alpha<\beta} \phi_\alpha) = \phi_\gamma.$$

$\leq_V$

For ϕ, ψ in Ψ, we write

$$\phi \leq_V \psi$$

to mean that, for some ordinals $\alpha \leq \beta$, $\phi \in \Psi_\alpha$, $\psi \in \Psi_\beta$ and

$$V_\alpha(\psi) = \phi$$

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\Psi^0_{\omega+2}$	$\Psi^1_{\omega+2}$	$\Psi^2_{\omega+2}$	$\Psi^3_{\omega+2}$	$\Psi^4_{\omega+2}$	$\Psi^5_{\omega+2}$	$\Psi^6_{\omega+2}$	$\ldots$
$\Psi^0_{\omega+1}$	$\Psi^1_{\omega+1}$	$\Psi^2_{\omega+1}$	$\Psi^3_{\omega+1}$	$\Psi^4_{\omega+1}$	$\Psi^5_{\omega+1}$	$\Psi^6_{\omega+1}$	$\ldots$
Ψ^0_{ω}	Ψ^1_{ω}	Ψ^2_{ω}	Ψ^3_{ω}	Ψ^4_{ω}	Ψ^5_{ω}	Ψ^6_{ω}	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
Ψ^0_2	Ψ^1_2	Ψ^2_2	Ψ^3_2	Ψ^4_2	Ψ^5_2	Ψ^6_2	$\ldots$
Ψ^0_1	Ψ^1_1	Ψ^2_1	Ψ^3_1	Ψ^4_1	Ψ^5_1	Ψ^6_1	$\ldots$
Ψ^0_0	Ψ^1_0	Ψ^2_0	Ψ^3_0	Ψ^4_0	Ψ^5_0	Ψ^6_0	$\ldots$

$\mathcal{I}_{m,n}$

For all $m \leq n < \omega$, we let $\mathcal{I}_{m,n}$ be the set of injections from m to n.

We let i_n the identity function on n.

Given a tuple

$$\bar{b} = \langle b_0, \ldots, b_{n-1} \rangle$$

and $j \in \mathcal{I}_{m,n}$, the subtuple of $\bar{b}$ corresponding to j is

$$\langle b_{j(0)}, \ldots, b_{j(m-1)} \rangle.$$

The horizontal projection function

One can recursively define a class-sized function H, whose domain consists of pairs

$$(\phi, j)$$

with $\phi \in \Psi^n$ and $j \in \mathcal{I}_{m,n}$ (for some $m \leq n$ in ω) such that whenever M is a τ-structure, α is an ordinal, $\bar{a}$ is an injective n-tuple from M and $j \in \mathcal{I}_{m,n}$,

$$H(\phi^M_{\bar{a},\alpha}, j) = \phi^M_{\bar{b},\alpha},$$

where $\bar{b}$ is the subtuple of $\bar{a}$ corresponding to j.

The definition of H

- When $\phi \in \Psi^n_0$, $H(\phi, j)$ is the conjunction of all conjuncts from ϕ whose variable indices are contained in the range of j, with $x_{j(i)}$ replaced by x_i for each such index i.
- When $\phi \in \Psi^n_{\alpha+1}$,

$$V_\alpha(H(\phi, j)) = H(V_\alpha(\phi), j)$$

and $E(H(\phi, j)) =$

$$\{H(\psi, j \cup \{(m, y)\}) \mid \psi \in E(\phi), y \in n + 1 \setminus \mathrm{range}(j)\}.$$

- When β is a limit ordinal

$$H(\bigwedge_{\alpha<\beta} \phi_\alpha, j) = \bigwedge_{\alpha<\beta} H(\phi_\alpha, j).$$

Scott Processes

A *Scott process* is a sequence $\langle \Phi_\alpha : \alpha < \delta \rangle$, for some ordinal δ, satisfying the following conditions.

The Formula Conditions

- Each Φ_α is a subset of Ψ_α, and, letting Φ^n_α denote $\Phi_\alpha \cap \Psi^n_\alpha$, each Φ^n_α is closed under permutations of X_n.
- For each ordinal of the form $\alpha + 1 < \delta$, and each $\phi \in \Phi_{\alpha+1}$, $E(\phi)$ is a subset of Φ_α.
- For all $\alpha < \beta < \delta$, $\Phi_\alpha = V_\alpha[\Phi_\beta]$.
- For all $\alpha < \delta$, and all $m < n$ in ω, $\Phi^m_\alpha = H[\Phi^n_\alpha \times \{i_m\}]$.

The Coherence Conditions

- For each ordinal of the form $\alpha + 1$ below δ, each $n \in \omega$ and each $\phi \in \Phi^n_{\alpha+1}$,

$$E(\phi) = V_\alpha[\{\psi \in \Phi^{n+1}_{\alpha+1} \mid H(\psi, i_n) = \phi\}].$$

- For all $\alpha < \beta < \delta$, all $n \in \omega$ and all $\phi \in \Phi^n_\beta$,

$$E(V_{\alpha+1}(\phi)) \subseteq V_\alpha[\{\psi \in \Phi^{n+1}_\beta \mid H(\psi, i_n) = \phi\}].$$

- For all $\alpha < \delta$, n, m in ω, $\phi \in \Phi^n_\alpha$ and $\psi \in \Phi^m_\alpha$, there exist $\theta \in \Phi^{n+m}_\alpha$ and $j \in \mathcal{I}_{m,n+m}$ such that $\phi = H(\theta, i_n)$ and $\psi = H(\theta, j)$.

Rank

The *rank* of a Scott process $\langle \Phi_\alpha : \alpha < \delta \rangle$ is the least α such that $\alpha + 1 < \delta$ and V_α is injective on $\Phi_{\alpha+1}$, if such an α exists.

When this happens, V_α is injective on Φ_β for all $\beta \in (\alpha, \delta)$.

If M is an infinite τ-structure, then, for any ordinal α, $\mathrm{SP}_\alpha(M)$ is a Scott process in the sense just defined.

The Scott rank of M is the rank of the corresponding Scott process of length $|M|^+$.

Models of processes

A τ-structure M is said to be a model of a Scott process P if $P = \mathrm{SP}_\alpha(M)$ for some ordinal α.

The Scott processes of countable length with all levels countable are exactly the initial segments of the Scott processes of countable τ-structures.

Question. Is the same true for $\aleph_1$?

No for $\aleph_2$: theorems of Laskowski-Shelah and Hjorth show that there is a Scott process of height ω_2 which has a model in a forcing extension but not in V.

Successor levels

Theorem
Every Scott process of countable (successor) length, with all levels countable, has a model.

Proof sketch

Suppose that the process has length $\delta + 1$. Letting the model have domain $\{c_n : n \in \omega\}$, assign each tuple $\bar{c} \restriction m = \langle c_n : n < m \rangle$ a formula ϕ_n from Φ^n_δ, in such a way that

$$H(\phi_{n+1}, i_n) = \phi_n.$$

$V_0(\phi)$ determines the truth values of the relations on $\bar{c} \restriction m$.

By suitable bookkeeping one can ensure that $\bar{c} \restriction m$ satisfies ϕ_n in the resulting τ-structure.

In the case where $\delta = \gamma + 1$, one need only to ensure that for each $\psi \in E(\phi_n)$ there is some $m \geq n$ such that

$$H(V_\gamma(\phi_m), i_n \cup (n, m+1)) = \psi.$$

Questions

How much control does one have on the resulting structure?

It is not hard to see when one can build a structure of Scott rank at most δ.

It is apparently harder to see when one can build a structure of larger rank.

Omitting a formula

Question Given a Scott process

$$P = \langle \Phi_\alpha : \alpha \leq \delta \rangle,$$

$n \in \omega$, and $\phi \in \Phi_\delta^{n+1}$, when do there exist $\Phi_{\delta+1}$ and $\psi \in \Phi_{\delta+1}^n$ such that

$$P^\frown \Phi_{\delta+1}$$

is a Scott process,

$$V_\delta(\psi) = H(\phi, i_n)$$

and

$$\phi \notin E(\psi)?$$

Amalgamation

Given an ordinal δ, a set $\Phi \subseteq \Psi_\delta$ *amalgamates* if for all $m \in \omega$ and all

$$\phi, \psi \in \Phi \cap \bigcup_{n \geq m} \Psi^n_\delta$$

if

$$H(\phi, i_m) = H(\psi, i_m)$$

then there exist $\theta \in \Phi$ and j_1, j_2 which are the identity on X_m such that

$$H(\theta, j_1) = \phi$$

and

$$H(\theta, j_2) = \psi.$$

A Scott process $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ extends to a Scott process

$$\langle \Phi_\alpha : \alpha \leq \delta + 1 \rangle$$

of rank at most δ if and only if Φ_δ amalgamates.

Then for all $n \in \omega$ and all $\phi \in \Phi^n_{\delta+1}$,

$$E(\phi) = \{\psi \in \Phi_\delta \mid H(\psi, i_n) = V_\delta(\phi)\}.$$

Building a model of cardinality $\leq \aleph_1$

(Harrington*) If $P = \langle \Phi_\alpha : \alpha \leq \delta \rangle$ is a Scott process such that $|\Phi_\delta| \leq \aleph_1$ and Φ_δ amalgamates, then there is a model of P of Scott rank at most δ and cardinality at most $\aleph_1$.

The corresponding statement is false when $|\Phi_\delta| \geq \aleph_2$.

Question In Harrington's theorem, if we drop the amalgamation condition, can we still build a model, possibly of Scott rank larger than δ? (We can weaken it slightly.)

Extending at limits

Theorem

If δ is a limit ordinal and

$$P = \langle \Phi_\alpha : \alpha < \delta \rangle$$

is a Scott process such that each Φ_α is countable, then there is a Scott process of length $\delta + 1$ extending P.

Proof sketch

Let $\langle \beta_n : n \in \omega \rangle$ be increasing and cofinal in δ, and choose formulas $\phi_n \in \Phi^n_{\beta_n}$ such that, for all n, $H(V_{\beta_n}(\phi_{n+1}), i_n) = \phi_n$.

For each $n \in \omega$, let ψ_n be

$$\bigwedge \{ V_\alpha(H(\phi_m, i_n)) : m \in \omega \setminus n, \alpha \leq \beta_m \}.$$

With sufficient bookkeeping, we can let Φ_δ be the set of formulas of the form

$$\{ H(\psi_n, j) : n \in \omega, j \in \bigcup_{k \leq n} \mathcal{I}_{k,n} \}.$$

Question. Does the thoerem hold for all Scott processes of limit length? (Yes for Scott processes derived from counterexamples to Vaught's conjecture.)

A *path* through a Scott process of limit length δ is a $\phi \in \Psi_\delta$ such that $V_\alpha(\phi) \in \Phi_\alpha$ for all $\alpha < \delta$.

Question If P is a Scott process of limit length, and Φ is the set of all paths through P, must $P^\frown\Phi$ be a Scott process? Must Φ amalgamate? Must Φ be nonempty? What if Φ contains a member $\leq_V$-above each member of each level of P?

Question. Suppose that M_α ($\alpha < \omega_1$) are countable τ-structures such that, for all $\alpha < \omega_1$ the process $\mathrm{SP}_\alpha(M_\beta)$ is the same for all $\beta \geq \alpha$. Must there be a τ-structure M such that

$$\mathrm{SP}_{\omega_1}(M) = \bigcup_{\alpha<\omega_1} \mathrm{SP}_\alpha(M_\alpha)?$$

Yes for countable limits instead of ω_1.

Scattered processes

A Scott process P is *scattered* if $2^{<\omega}$ does not embed into the restriction of $\leq_V$ to the formulas in P.

A Scott process of countable limit length is scattered if all levels are countable and the process has fewer than continuum many models (e.g., a counterexample to Vaught's Conjecture).

Isolated paths

Let $P = \langle \Phi_\alpha : \alpha < \delta \rangle$ be a Scott process of limit length.

A path ϕ is *isolated* if for some $\alpha < \delta$, ϕ is the unique path in

$$V_\alpha^{-1}[\{V_\alpha(\phi)\}] \cap \Psi_\delta.$$

If P is scattered, then, letting Φ_I be the set of isolated paths through P,

$$P^\frown \Phi_I$$

is a Scott process with the property that every proper extension has rank δ.

Minimal sets

If P is a scattered Scott process of limit length δ there is, for each path θ through P a smallest set $\Phi(\theta)$ for which $P^\frown\Phi(\theta)$ is a Scott process.

Question Must $\Phi(\theta)$ amalgamate? (Yes, if θ is isolated.)

Absoluteness

The following theorem follows from Σ^1_1 absoluteness and a theorem of Solovay saying that the intersection of mutually generic forcing extensions is the ground model.

Theorem

Suppose that $\mathcal{A} \subseteq \mathcal{S}_\tau$ is a counterexample to the analytic Vaught conjecture, and let $x \subseteq \omega$ be such that $\mathcal{A}$ is Σ^1_1 in x. Then for any ordinal α and any model M of $\mathcal{A}$ in any forcing extension, $\mathrm{SP}_\alpha(M) \in L[x]$.

Since the ranks of the Scott processes of models of a counterexample to the analytic Vaught conjecture are cofinal in ω_1, the theorem implies that the ranks of the correspoinding Scott processes are cofinal in each regular cardinal of $L[x]$.

Theorem (Sacks*)

If $\phi \in \mathcal{L}_{\aleph_1,\aleph_0}$ is a counterexample to Vaught's Conjecture, then for club many δ below each of ω_1 and ω_2, ϕ has distinct models M and N of Scott rank δ such that

$$\mathrm{SP}_\delta(M) = \mathrm{SP}_\delta(N),$$

Φ^M_δ is the set of isolated paths through SP^M_δ and Φ^N_δ is the set of all paths through SP^N_δ.

Question. Must a counterexample to Vaught's Conjecture have three nonisomorphic models of the same rank?

Amalgamation past ρ

Given an ordinal δ, $k \in \omega$ and $\rho \in \Psi_\delta^m$, a set $\Phi \subseteq \Psi_\delta$ *amalgamates past* ρ if for all $m \geq k$ and all

$$\phi, \psi \in \Phi \cap \bigcup_{n \geq m} \Psi_\delta^n,$$

if

$$H(\phi, i_m) = H(\psi, i_m)$$

and

$$H(\phi, i_k) = \rho$$

then there exist $\theta \in \Phi$ and j_1, j_2 which are the identity on X_m such that

$$H(\theta, j_1) = \phi$$

and

$$H(\theta, j_2) = \psi.$$

Given a Scott process $P = \langle \Phi_\alpha : \alpha \leq \delta + 1 \rangle$ and a $\rho \in \Phi^k_\delta$, we say that P is injective past ρ if for all $\phi \in \Phi^m_\delta$ with $m \geq k$ and $H(\phi, i_k) = \rho$,

$$|V_\delta^{-1}[\{\phi\}] \cap \Phi_{\delta+1}| = 1.$$

Every model of such a process has Scott rank at most $\delta + k$.

If $P = \langle P_\alpha : \alpha \leq \delta \rangle$ is such that Φ_δ amalgamates past ρ, then P has an extension of length $\delta + 1$ which is injective past ρ.

Theorem

Suppose that $P = \langle \phi_\alpha : \alpha \leq \delta \rangle$ *is a Scott process with* Φ_δ *countable, and that* Φ_δ *does not amalgamate past any of its members. Then there are continuum many nonisomorphic models of* P*.*

Proof sketch

There exists a $2^{<\omega}$-tree of formulas in Φ_δ, ordered by H with the identity function, such that each pair of immediate successors witnesses a failure of amalgamation past their common predecessor (and also meets some bookkeeping condition).

The paths through the tree correspond to models of the process.

The root of the tree corresponds to a formula ϕ. In each model of the process, each tuple satisfying ϕ corresponds to at most one path through the tree.

So no model can represent uncountably many paths.

It follows that if $\psi \in \mathcal{L}_{\aleph_1,\aleph_0}$ is a counterexample to Vaught's Conjecture, then the Scott ranks of the models of ψ include γ and infinitely members of each interval $[\gamma, \gamma + \omega)$, whenever $\gamma < \omega_2$ is a limit ordinal greater than the quantifier depth of ψ.

Question. Do they in fact include every member of $(\mathrm{qd}(\psi), \omega_2)$?

More on the number of models

Given an analytic set $\mathcal{A} \subseteq \mathcal{S}_\tau$, we let $\mathcal{A}^*$ denote the class of (ground model, but possibly uncountable) τ-structures M which are isomorphic to an element of the reinterpretation of $\mathcal{A}$ in any (equivalently, every, by Σ^1_1-absoluteness) outer model in which M is countable.

If $\mathcal{A}$ is the set of τ-structures on ω satisfying a sentence ϕ of $\mathcal{L}_{\aleph_1,\aleph_0}(\tau)$, then $\mathcal{A}^*$ as defined above is simply the class of models of ϕ.

For an ordinal α, we let $\mathrm{SP}_\alpha(\mathcal{A})$ denote the set of the Scott processes of length α for structures in $\mathcal{A}^*$.

We also let $\mathcal{A}_\alpha$ denote respectively the class of structures $\mathcal{A}^*$ of Scott rank α.

The following is an alternate version of the absoluteness fact stated above.

Theorem

Suppose that $\mathcal{A}$ is a counterexample to the analytic Vaught conjecture, and let $x \subseteq \omega$ be such that $\mathcal{A}$ is Σ^1_1 in x. Let Y be a countable elementary submodel of $H((2^{\aleph_1})^+)$ with $x \in Y$, let

$$\delta = Y \cap \omega_1$$

and let P be the transitive collapse of Y. Then

$$\mathrm{SP}_{\delta+1}(\mathcal{A}) = \mathrm{SP}_{\delta+1}(\mathcal{A})^P.$$

Theorem (Larson-Shelah)

Suppose that $\mathcal{A}$ is a counterexample to the analytic Vaught Conjecture and

$$\gamma \in \omega \cup \{\aleph_0\}$$

is such that there are up to isomorphism exactly γ many elements of $\mathcal{A}^*$ of Scott rank ω_1. Then for club many $\alpha < \omega_1$ there are exactly γ many models in $\mathcal{A}$ of Scott rank α, up to isomorphism.

Proof: Let $\mathcal{M} = \{M_n : n \leq \gamma\}$ be pairwise nonisomorphic elements of $\mathcal{A}_{\omega_1}$ such that every element of $\mathcal{A}_{\omega_1}$ is isomorphic to some element of $\mathcal{M}$. Let $\mathcal{Y}$ be the set of countable elementary substructures of $H((2^{\aleph_1})^+)$ containing (as elements) $\mathcal{M}$ and a (fixed) code for $\mathcal{A}$.

We show that for each $Y \in \mathcal{Y}$, letting $\mathcal{M}_Y$ be the image of $\mathcal{M}$ under the transitive collapse of Y, every element of $\mathcal{A}_{Y \cap \omega_1}$ is isomorphic to an element of $\mathcal{M}_Y$. As the members of $\mathcal{M}_Y$ will be nonisomorphic, this will establish the theorem.

Fix $Y \in \mathcal{Y}$, let $\alpha = Y \cap \omega_1$ and let P be the transitive collapse of Y. Then

$$\mathrm{SP}_{\alpha+1}(\mathcal{A}) = \mathrm{SP}_{\alpha+1}(\mathcal{A})^P.$$

Suppose toward a contradiction that there exists an

$$N \in \mathcal{A}_\alpha \setminus \mathcal{M}_Y.$$

Then $\mathrm{SP}_{\alpha+1}(N) \in P$. Then the δ-th level of $\mathrm{SP}_{\delta+1}(N)$ amalgamates. Since amalgamation is a first order property it is witnessed in P. It follows that there is a model of $\mathrm{SP}_{\delta+1}(N)$ in P, contradicting the elementarity of the collapse and the assumed property of $\mathcal{M}$. $\square$