Strongly unbounded colorings

(joint work with Assaf Rinot)

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Kobe Set Theory Workshop
on the occasion of Sakaé Fuchino's retirement

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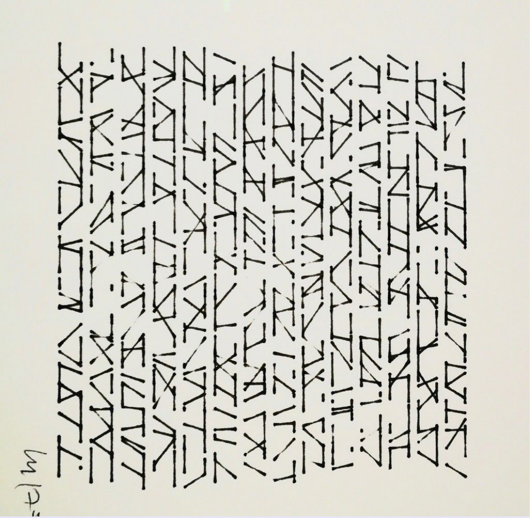
Productivity of chain conditions



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The κ-Knaster property is a strengthening of the κ-cc.

In contrast to the κ-cc, the κ-Knaster property is always *productive*: if $\mathbb{P}$ and $\mathbb{Q}$ are κ-Knaster, then $\mathbb{P} \times \mathbb{Q}$ are κ-Knaster.

Infinite productivity

For an infinite cardinal θ, we say that the κ-Knaster property is *θ-productive* if, whenever $\{\mathbb{P}_i \mid i < \theta\}$ are all κ-Knaster, the full-support product $\prod_{i<\theta} \mathbb{P}_i$ is κ-Knaster.

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Theorem (LH-Lücke, '18)

If the κ-Knaster property is $\aleph_0$-productive, then κ is weakly compact in L.

Accessible cardinals

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Suppose that κ is a successor cardinal. Then there is a κ-Knaster poset $\mathbb{P}$ such that $\mathbb{P}^{\aleph_0}$ is not κ-cc.

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The proof of this theorem involved colorings $c : [\kappa]^2 \to \omega$ with strong unboundedness properties and initiated a systematic investigation of such colorings, their variations, and their applications.

References

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[2] ______, *Knaster and friends II: The C-sequence number*, J. Math. Log. **21** (2021), no. 01, 2150002.

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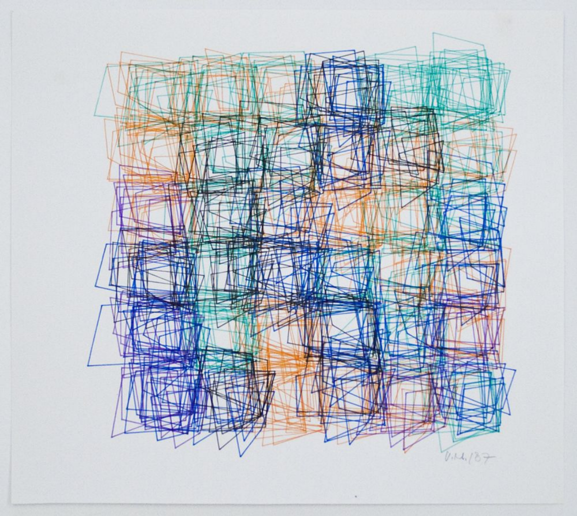
Chapter 1

Strongly unbounded colorings



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Note that, for all $\mu' \leq \mu$ and $\chi' \leq \chi$, $\mathrm{U}(\kappa, \mu, \theta, \chi)$ implies $\mathrm{U}(\kappa, \mu', \theta, \chi')$, but there is no such obvious monotonicity in the third coordinate.

Failure of infinite productivity

Lemma

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Let $c : [\kappa]^2 \to \theta$ witness $\mathrm{U}(\kappa, \kappa, \theta, \chi)$.

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Further proof sketch?

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Theorem (Todorcevic)

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Corollary

For every successor cardinal κ, the κ-Knaster property fails to be $\aleph_0$-productive.

Provable instances of $\mathrm{U}(\kappa, \mu, \theta, \chi)$

Given a coloring $c : [\kappa]^2 \to \theta$, an ordinal $\beta < \kappa$, and a color $i < \theta$, let $D^c_{\leq i}(\beta) := \{\alpha < \beta \mid c(\alpha, \beta) \leq i\}$.

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Moreover, in all instances, $\mathrm{U}(\kappa, \kappa, \theta, \chi)$ is witnessed by a closed coloring.

Consistent failures of $\mathrm{U}(\kappa, \mu, \theta, \chi)$

Proposition (LH-Rinot, [1])

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2. *If there is a supercompact cardinal, then there is a forcing extension in which $\mathrm{U}(\aleph_{\omega+1}, 2, \aleph_k, \aleph_1)$ fails for all $1 \leq k < \omega$.*

Chapter 3

The C-sequence number

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Definition (The C-sequence number)

For every regular, uncountable cardinal κ, let $\chi(\kappa) = 0$ if κ is weakly compact. Otherwise, let $\chi(\kappa)$ be the least cardinal χ such that, for every C-sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$, there is an unbounded $D \subseteq \kappa$ such that, for every $\alpha < \kappa$, there is $b \in [\kappa]^\chi$ for which $D \cap \alpha \subseteq \bigcup_{\beta \in b} C_\beta$.

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Some consistency results

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Note the similarity to the previous consistency result about failures of $\mathrm{U}(\kappa, \mu, \theta, \chi)$.

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Theorem (LH-Rinot, [2])

1. *Suppose that κ is weakly compact.*
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1. *there is a C-sequence $\vec{C}$ over κ such that $\chi(\vec{C}) = \theta$;*
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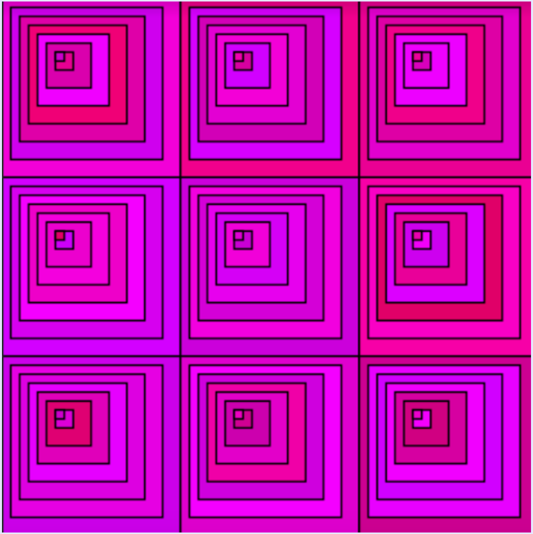
   - $\mathrm{U}(\kappa, 2, \aleph_0, 2)$ *fails.*

Chapter 3

Subadditive colorings



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Suppose that $\square(\kappa)$ *holds. Then, for every regular* $\theta < \kappa$*, there is a subadditive witness to* $\mathrm{U}(\kappa, \kappa, \theta, \chi)$ *for all* $\chi < \kappa$*.*

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Some recent work has been done studying the relationship between $t(X)$ and $t(X_\delta)$. Of particular interest is whether there is an upper bound on $t(X_\delta)$ for countably tight (or stronger) spaces X.

Some results

Theorem (Dow-Juhász-Soukup-Szentmiklóssy-Weiss, '19)

If there is a non-reflecting stationary subset of $\kappa \cap \mathrm{cof}(\omega)$, then there is a Fréchet space X such that $t(X_\delta) = \kappa$.

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If PID holds and X is a Fréchet α_1-space, then $t(X_\delta) \leq \aleph_1$.

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Suppose that μ is a singular cardinal of countable cofinality and SCH fails at μ. Then there is a Fréchet α_1-space X such that $t(X_\delta) = \mu^+$.

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- *c is locally small, i.e., $|D^c_{\leq i}(\beta)| < \mu$ for all $i < \mathrm{cf}(\mu)$ and all $\beta < \mu^+$.*

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i.e., the closed subsets of μ^+ are precisely the intersections of the sets $D^c_{\leq i}(\beta)$, so, if $T \subseteq \mu^+$, then

$$\infty \in \mathrm{cl}(T) \Leftrightarrow (\nexists (i,\beta) \in \omega \times \mu^+)(T \subseteq D^c_{\leq i}(\beta))$$

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Suppose now that $|T| \geq \mu$. For each (i, β), we have $|D^c_{\leq i}(\beta)| < \mu$, so $|[D^c_{\leq i}(\beta)]^{\aleph_0}| < \mu$. Therefore, the number of elements of $[T]^{\aleph_0}$ that are contained in some $D^c_{\leq i}(\beta)$ is at most μ^+. Since SCH fails at μ, $|[T]^{\aleph_0}| > \mu^+$, so we can find a countable subset of T that is not contained in any $D^c_{\leq i}(\beta)$, and proceed as in the previous case.

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However, for every $\beta < \mu^+$, we have $\beta = \bigcup_{i<\omega} D^c_{\leq i}(\beta)$. Since $D^c_{\leq i}(\beta)$ is closed in X, β is closed in X_δ. Therefore, ∞ is not in the closure of any bounded subset of μ^+ in X_δ. It follows that $t(X_\delta) = \mu^+$.

All artwork by Vera Molnár.

Thank you!