Measurable cardinals and limits in the category of sets

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Sakaé Fuchino's Retirement Conference
10 March 2021

New results are joint work with Jiří Adamek, Tim Campion, Leonid Positselski and Jiří Rosický.

Category theory preliminaries

Recall

A category $\mathcal{C}$ consists of

- a class of *objects*, and
- for every pair of objects A and B of $\mathcal{C}$, a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of *morphisms* f from A to B, written $f : A \to B$,
- for every object A, a distinguished morphism 1_A in $\mathrm{Hom}_{\mathcal{C}}(A, A)$,
- a *composition* function $\circ$, taking $f : A \to B$ and $g : B \to C$ to $g \circ f : A \to C$

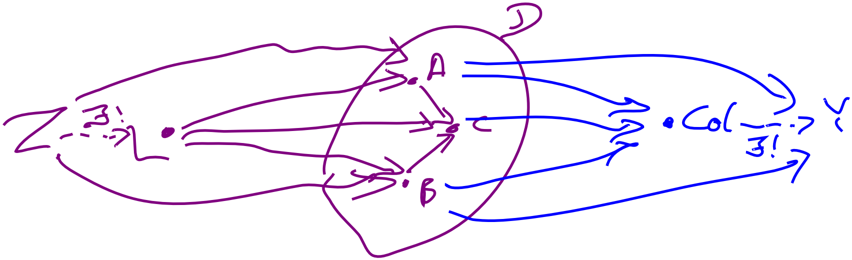
such that composition is associative, and for any $f : A \to B$, $f \circ 1_A = f = 1_B \circ f$.

E.g.s

- **Set** is the category with sets as objects and functions as morphisms.
- **Gp** is the category with groups as objects and group homomorphisms as morphisms.

We think of a diagram as being a set of objects and morphisms between them.

- The limit of a diagram $\mathcal{D}$ is an object L along with a *cone* δ of projection maps to the objects of $\mathcal{D}$ (such that the triangles formed with the morphisms of $\mathcal{D}$ commute) such that any other such cone from an object of $\mathcal{C}$ factors uniquely through δ.
- The colimit of a diagram is the same in reverse.



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In **Set**, every diagram $\mathcal{D}$ has a limit and a colimit:

- The limit is the subset of the product of the sets in $\mathcal{D}$ consisting of all element whose coordinates "cohere" under the functions of the diagram.
- The colimit is the disjoint union of the sets in $\mathcal{D}$, modulo identifying elements with their images under the functions in $\mathcal{D}$.

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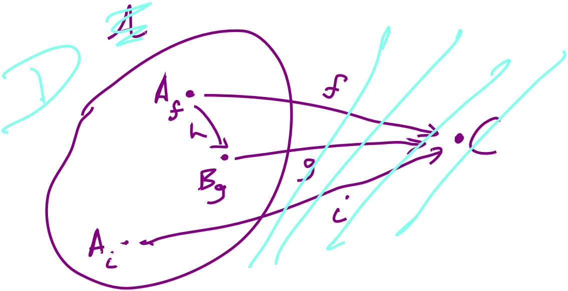
Gp has all limits & colimits too: limits are the same as in **Set**, and colimits are free products modulo identifications.

Given a set $\mathcal{A}$ of objects in a category $\mathcal{C}$ and an object C of $\mathcal{C}$, the canonical diagram of C with respect to $\mathcal{A}$ is the diagram with

- for every object A in $\mathcal{A}$ and every morphism $f\colon A \to C$, a copy of A, which we shall denote by A_f,
- as morphisms, all morphisms $h\colon A_f \to B_g$ such that $g \circ h = f$.



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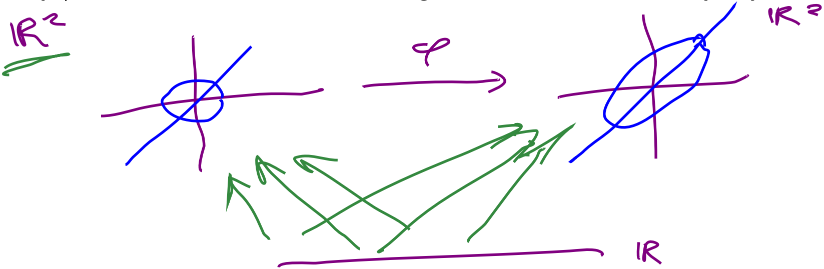
E.g.s

- ω is dense in **Set**: every set is the colimit of the diagram of all of its finite subsets, which are the images of functions from finite sets.
- Any set of representatives of all the isomorphism classes of finitely generated groups is dense in **Gp**: every group is the colimit of the diagram of all of its finitely generated subgroups.

Being a canonical colimit of objects from $\mathcal{A}$ is stronger in general than just being a colimit of *some* diagram of objects from $\mathcal{A}$.

E.g.

Let $\mathbf{Vect}_{\mathbb{R}}$ be the category of real vector spaces, with linear transformations as the morphisms. Consider $\mathcal{A} = \{\mathbb{R}\}$. Then every object of $\mathbf{Vect}_{\mathbb{R}}$ is a colimit of objects from $\mathcal{A}$, but $\mathcal{A}$ is *not* dense. Indeed, consider a function $\varphi \colon \mathbb{R}^2 \to \mathbb{R}^2$ respecting scalar multiplication but not addition. Then there is a cocone mapping each $\mathbb{R}_f$ to $\mathbb{R}^2$ by $\varphi \circ f$, but it doesn't factor through the canonical cocone by any linear map.



Opposite categories

Given a category $\mathcal{C}$, $\mathcal{C}^{op}$ is the category with the same objects as $\mathcal{C}$, and the same morphisms but in the opposite direction. Identity functions remain identity functions, and compositions of morphisms remain compositions of morphisms, just in the opposite order.

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$\mathbf{Set}^{op}$ is the category with sets as objects, and functions as morphisms, with any $f : X \to Y$ in the usual sense being considered as going from Y to X.

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Question

Is there a dense set in $\mathbf{Set}^{op}$?

For any cardinal κ and any set X, consider the canonical diagram $\mathcal{D}$ in $\mathbf{Set}^{op}$ of X with respect to κ.

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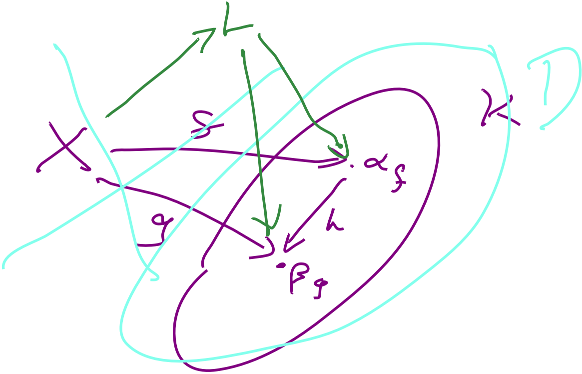

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We can think about such functions $f : X \to \alpha$ in terms of the partitions $\{f^{-1}\{\gamma\} \mid \gamma \in \alpha\}$ that they define. In this context, the functions in the diagram represent coarsening maps.

The elements of the limit are elements $\tilde{\mu} = (\mu_f)_{\alpha_f \in \mathcal{D}}$ of the product of the ordinals α_f in $\mathcal{D}$ — in the α_f coordinate, the element μ_f of α_f is chosen.

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This corresponds to the choice of a piece from each of the partitions $(f^{-1}\{\mu_f\}$ in the partition corresponding to $f\colon X \to \alpha)$, in a way that the coarsening maps respect — we can think of this as choosing a "big" piece from each partition.

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$\mathcal{U}$ is a κ-complete ultrafilter

Let $\chi_Y \colon X \to 2$ be the characteristic function of Y, $\chi_Y(x) = 1 \leftrightarrow x \in Y$. Then

$$\begin{aligned}\mathcal{U} &= \{Y \subseteq X \mid \exists \alpha < \kappa \exists f \colon X \to \alpha (Y = f^{-1}\{\mu_f\})\} \\ &= \{Y \subseteq X \mid \mu_{\chi_Y} = 1\}.\end{aligned}$$

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- For any $Y \subseteq X$, $Y \in \mathcal{U}$ if $u_{\chi_Y} = 1$ and $X \smallsetminus Y \in \mathcal{U}$ if $u_{\chi_Y} = 0$, so $\mathcal{U}$ is ultra.

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Note that by definition, there is a dense set in $\mathbf{Set}^{op}$ if and only if for some κ, every X is the limit of its canonical diagram with respect to κ, if and only if there are no non-principal κ-complete ultrafilters on any set.

So we have shown

Theorem (Isbell, 1960)

There is a dense set in $\mathbf{Set}^{op}$ *if and only if there are only boundedly many measurable cardinals.*

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On the other hand $\{\mathbb{R}^2\}$ *is* dense in $\mathbf{Vect}_{\mathbb{R}}$.

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Proof

By case distinction on whether there is a proper class of measurable cardinals!

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- Consider the category of structures (X, Y, E) for the language with one unary predicate and one binary relation, such that the field of E is contained in Y (i.e. (Y, E) is a directed graph). As morphisms in the category, take all functions $f\colon (X_0, Y_0, E_0) \to (X_1, Y_1, E_1)$ such that $f \restriction Y_0$ is a graph homomorphism from (Y_0, E_0) to (Y_1, E_1), and such that f maps each element of $X_0 \smallsetminus Y_0$ either to an element of $X_1 \smallsetminus Y_1$, or to an element of Y_1 in the image of some homomorphism $G \to (Y_1, E_1)$ with $G \in \mathcal{G}$.

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- $\{(1, 1, \emptyset), (2, 2, \{(0, 1)\}), (1, \emptyset, \emptyset)\}$ is colimit dense in this category, but a short argument shows that a dense set would give rise to homomorphisms between members of $\mathcal{G}$.

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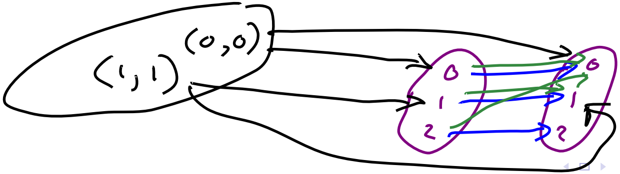
Proof of Lemma

First, for $n = 0, 1$, or 2 (or indeed 3), an n-element set is the limit of the diagram

$$3 \underset{f_n}{\overset{1_3}{\rightrightarrows}} 3$$

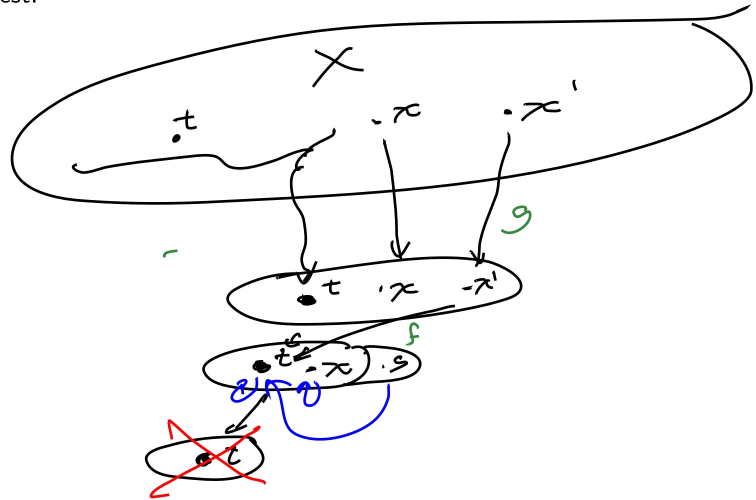
where

$$f_n(i) = \begin{cases} i & \text{if } i < n \\ i + 1 \pmod 3 & \text{otherwise.} \end{cases}$$



Idea for sets of cardinality ≥ 2:

Instead of the partition perspective, think of stripping off individual elements from "the rest."



Done formally:

Suppose X with cardinality at least 2 is given. Choose $t \in X$, and take $s \notin X$. For each $x \in X \setminus \{t\}$, let

$$K_x = \{t, x, s\}.$$

For any set Y containing x, define $f_{Y,x} \colon Y \to K_x$ by

$$f_{Y,x}(a) = \begin{cases} x & \text{if } a = x \\ t & \text{otherwise.} \end{cases}$$

Let $\mathcal{D}$ be the diagram with objects all the sets K_x and all the 3-element subsets of X containing t, and with morphisms all the functions $f_{Y,x} \colon Y \to K_x$ for Y a 3-element subset of X, and all the functions $f_{K_x,x} \colon K_x \to K_x$.

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Then X is the limit in **Set** of $\mathcal{D}$, with limit maps $f_{X,x} \colon X \to K_x$, and similarly $g_{X,x,y} \colon X \to \{t, x, y\}$ defined by

$$g_{X,x,y}(a) = \begin{cases} a & \text{if } a = x \text{ or } a = y \\ t & \text{otherwise.} \end{cases}$$