On forcing names for ultrafilters

Piotr Borodulin-Nadzieja (University of Wrocław)

Kobe Set Theory Workshop 2021

on the occasion of Sakaé Fuchino's Retirement





An artefact from my office

Countable Chain Condition の Variations に関する

リマー

渕野 昌

(Freie Universität Berlin)

以下では B をブール代数とする。B+'で、B の 0 と異なる元の全体を表わすことにする。 B の任意の非可算部分集合 X に対し、 $x,y \in X$ で、 $x \neq y$ かつ、 $x \cdot y \neq 0$ となるものが存在 するとき B は ccc (countable chain condition)を満たすという. V を集合論の universe として、V の中でのブール代数 B が ccc を満たすとする. G を B^+ (あるいは B の V で の完備化) に関する V 上の generic set とすれば、よく知られているように、V の基数は V[G] でも基数として保存されている。この性質のために、ccc を満たすブール代数のクラ スは、強制法の理論で、きわめて重要な役割をはたす。しかしながら、ccc を満たすブール 代数の具体的な例について、それが ccc を満たしていることの証明を詳しく調べてみると、 同じ証明で、実は B が ccc より更に強い条件を満たしていることが示せている場合が多い。 以下では、そのような条件のうち、 σ-centred および all product ccc と呼ばれるものについ ての特徴付けを与える 命題 1,2 を証明する. ccc に関連する,他の条件や,これらの担互関 係などについては、[1] や [7] などを参照されたい. 命題 1 は、一年ほど前に、S.Koppelberg 教授に教えていただいたものである。これらの結果は、elementary で、多分 forklore に属す ものと思われるが、特に 命題1 は集合論的トポロジーの方での連鎖条件の研究者が見落と していることが多いように見受けられるので、一度書き出しておく価値があるのではない かと思う。

命題2から先では、forcing に関するごく初等的な知識を仮定している。これに関しては [5]

Based on ...

The talk is based on two preprints:

- PBN, Damian Sobota, On sequences of homomorphisms into measure algebras and the Efimov Problem (arxiv)
- PBN, Katarzyna Cegiełka, On measures induced by forcing names for ultrafilters

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Applications:

- reals are ultrafilters on the Cantor algebra,
- Stone spaces of old Boolean algebras may provide interesting examples of topological spaces.

A name induces a homomorphism

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$$\varphi(A) = \|A \in \dot{u}\|.$$

Then φ is a Boolean homomorphism.

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Let $\varphi \colon \mathbb{A} \to \mathbb{P}$ be a Boolean homomorphism. Consider a \mathbb{P} -name \dot{u} defined by

$$\dot{\varphi} = \{ \langle A, \varphi(A) \rangle \colon A \in \mathbb{A} \}.$$

Then $\dot{\varphi}$ is a name for an ultrafilter on \mathbb{A} .

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For every \mathbb{P} -name \dot{u} for an ultrafilter on \mathbb{A} there is a Boolean homomorphism $\varphi \colon \mathbb{A} \to \mathbb{P}$ such that

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Remark. Ultrafilters = homomorphisms to $\{0, 1\}$.

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For a cardinal number κ define the measure algebra of type κ by

$$\mathbb{M}_{\kappa} = \mathrm{Bor}(\{0,1\}^{\kappa})_{/\lambda_{\kappa}=0}.$$

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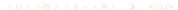
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- \blacksquare $\mathbb{M}_{\omega} = \operatorname{Bor}[0,1]_{/\mathcal{N}}$,
- Forcing with $\mathbb{M}_{\kappa} = \text{adding } \kappa \text{ random reals (for } \kappa > \omega).$



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Remark. By a "measure" we mean here a *finitely additive* measure.



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Suppose that $\mu = \delta_u$ for some ultrafilter u on \mathbb{A} (in V!). Then . . .

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lacksquare If μ is purely atomic, then

$$1 \Vdash \dot{\varphi} \in V$$
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- if μ is the standard Lebesgue measure, then $\dot{\varphi}$ is a "random" real.
- lacksquare if μ is non-atomic, then $\dot{\varphi}$ is a "new" real.

Application: "reals".

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In the classical random model there are no well ordered chains of size ω_2 in $\mathcal{P}(\omega)/\text{Fin}$.

Proposition

Assume GCH. Suppose $(\varphi_{\alpha})_{\alpha<\omega_2}$ is a sequence of homomorphisms $\varphi_{\alpha}\colon \mathbb{C}\to \mathbb{M}_{\omega_2}$. Then there is an automorphisms $\Phi\colon \mathbb{M}_{\omega_2}\to \mathbb{M}_{\omega_2}$ such that $\Phi\circ\varphi_{\alpha}=\varphi_{\beta}$ and $\Phi\circ\varphi_{\beta}=\varphi_{\alpha}$.

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- But this means that

$$\mathbb{M}_{\omega_2} \Vdash \dot{\varphi}_{\beta} \subseteq^* \dot{\varphi}_{\alpha}.$$



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• We say that α and β are symmetric if for each chunks A, B

$$\varphi_{\alpha}(A) \wedge \varphi_{\beta}(B) = 0 \iff \varphi_{\beta}(A) \wedge \varphi_{\alpha}(B) = 0$$



Claim. There are $\alpha < \beta < \omega_2$ which are symmetric, i.e.

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- Then" for each $\alpha < \beta < \omega_1$ we have $\varphi_{\alpha}(A) \wedge \varphi_{\beta}(B) = 0$ and $\varphi_{\beta}(A) \wedge \varphi_{\alpha}(B) \neq 0$.



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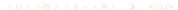
- For each $\alpha < \beta < \omega_1$ we have $\varphi_{\alpha}(A) \wedge \varphi_{\beta}(B) = 0$ and $\varphi_{\beta}(A) \wedge \varphi_{\alpha}(B) \neq 0$.
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- Then $D \wedge \varphi_{\gamma}(B) = 0$.

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- By ccc there is $\gamma < \omega_1$ such that $D = \bigvee_{\alpha < \gamma} \varphi_{\alpha}(A)$.
- Then $D \wedge \varphi_{\gamma}(B) = 0$.
- But $\varphi_{\gamma+1}(A) \wedge \varphi_{\gamma}(B) \neq 0$ and $\varphi_{\gamma+1}(A) \leq D$. A contradiction.



Claim. There are α and $\beta < \omega_2$ which are symmetric, i.e.

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So, by Sikorski's Extension Lemma, there is an automorphism $\Phi\colon \mathbb{M}\to \mathbb{M}$ such that

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and we are done.



Theorem (Kunen)

Application: peculiar topological spaces

Efimov problem

Theorem (Dow, Fremlin)

In the classical random model there is a Efimov space, i.e. an infinite compact space without a nontrivial convergent sequences and without a copy of $\beta\omega$.

Frechet-Nikodym metric

Definition

If $\mathbb M$ is a measure algebra, then $d_\lambda\colon \mathbb M \to [0,\infty)$ defined by

$$d_{\lambda}(A,B) = \mu(A \triangle B)$$

is a metric (called Frechet-Nikodym metric).

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We say that a sequence of homomorphisms $\varphi_n \colon \mathbb{A} \to \mathbb{M}$ converges metrically pointwise to a homomorphism $\varphi \colon \mathbb{A} \to \mathbb{M}$ if

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Remark. Stone topology = pointwise convergence topology.

Pointwise convergence and convergence in the extension

Proposition (PBN, Sobota)

Suppose that

 $\mathbb{M} \Vdash (\dot{\varphi}_n)$ converges to $\dot{\varphi}$.

Then (φ_n) converges metrically pointwise to φ .

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Proposition (PBN, Sobota)

Let M be a measure algebra. Then

$$\mathbb{M} \Vdash (\dot{\varphi}_n)$$
 converges to $\dot{\varphi}$

if and only if (φ_n) converges to φ pointwise algebraically.

Uniform convergence

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We say that a sequence of homomorphisms $\varphi_n \colon \mathbb{A} \to \mathbb{M}$ converges uniformly to a homomorphism $\varphi \colon \mathbb{A} \to \mathbb{M}$ if

$$\forall \varepsilon > 0 \ \exists N \ \forall n > N \ \forall A \in \mathbb{A} \ d_{\lambda}(\varphi_n(A), \varphi(A)) < \varepsilon.$$

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Theorem (PBN, Sobota, 2020)

Let \mathbb{M} be a measure algebra. If $\mathbb{M} \Vdash (\dot{\varphi}_n)$ converges trivially to $\dot{\varphi}$, then (φ_n) converges to φ uniformly.

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Let \mathbb{M} be a measure algebra. If $\mathbb{M} \Vdash (\dot{\varphi}_n)$ converges trivially to $\dot{\varphi}$, then (φ_n) converges to φ uniformly.

If $\mathbb{M} \Vdash (\dot{\varphi}_n)$ converges non-trivially to $\dot{\varphi}$, then (φ_n) does not converge to φ uniformly.



Assume $1 \Vdash \dot{\varphi} \neq \dot{\psi}$.

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. Then

$$1 \Vdash \exists A \in \mathbb{A} \ A \in \dot{\varphi} \setminus \dot{\psi}.$$

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There is an antichain (p_n) in \mathbb{M} and a sequence (A_n) of elements of \mathbb{A} such that

$$p_n \Vdash A_n \in \dot{\varphi} \setminus \dot{\psi}.$$

Assume $1 \Vdash \dot{\varphi} \neq \dot{\psi}$. Then

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There is an antichain (p_n) in \mathbb{M} and a sequence (A_n) of elements of \mathbb{A} such that

$$p_n \Vdash A_n \in \dot{\varphi} \setminus \dot{\psi}.$$

Can we say anything about $\lambda(p_n)$?



A fence exercise



Fence exercise

Shimamura and Komako are painting a picket fence between their properties. The fence consists of *n* many rails. Each rail has two sides—Shimamura's and Komako's—and those sides have to be painted in such a way that Shimamura's side has different colour than Komako's.

Fence exercise

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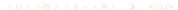
No matter how many colours they use, there is always a set B of at least n/4 many rails with the following property: the set of Shimamura's colours used in B is disjoint with the set of Komako's colours used in B.

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[I will present a proof by Dominik Gdesz]



The 1/4 theorem

Theorem (PBN, Sobota)

Let M be a measure algebra. If

$$\mathbb{M}\Vdash \exists A\in \mathbb{A}\ A\in \dot{\varphi}\setminus \dot{\psi},$$

then there is $p \in \mathbb{M}$, $A \in \mathbb{A}$ such that $\lambda(p) > 1/4$ and

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Theorem (PBN, Cegiełka)

For each $\kappa \geq \omega$ and a Boolean algebra $\mathbb A$ there is a sequence of homomorphisms from $\mathbb A$ into $\mathbb M_\kappa$ which converges pointwise metrically but not uniformly.

Thank you

