On forcing names for ultrafilters

Piotr Borodulin-Nadzieja (University of Wrocław)

Kobe Set Theory Workshop 2021

on the occasion of Sakaé Fuchino's Retirement

An artefact from my office

Countable Chain Condition の Variations に関する
リマーク

渕野　昌
(Freie Universität Berlin)

以下では B をブール代数とする. B^+ で, B の 0 と異なる元の全体を表わすことにする. B の任意の非可算部分集合 X に対し, $x, y \in X$ で, $x \neq y$ かつ, $x \cdot y \neq 0$ となるものが存在するとき B は ccc (countable chain condition) を満たすという. V を集合論の universe として, V の中でのブール代数 B が ccc を満たすとする. G を B^+ (あるいは B の V での完備化) に関する V 上の generic set とすれば, よく知られているように, V の基数は $V[G]$ でも基数として保存されている. この性質のために, ccc を満たすブール代数のクラスは, 強制法の理論で, きわめて重要な役割をはたす. しかしながら, ccc を満たすブール代数の具体的な例について, それが ccc を満たしていることの証明を詳しく調べてみると, 同じ証明で, 実は B が ccc より更に強い条件を満たしていることが示せている場合が多い. 以下では, そのような条件のうち, σ-centred および all product ccc と呼ばれるものについての特徴付けを与える **命題** 1, 2 を証明する. ccc に関連する, 他の条件や, これらの相互関係などについては, [1] や [7] などを参照されたい. **命題** 1 は, 一年ほど前に, S.Koppelberg 教授に教えていただいたものである. これらの結果は, elementary で, 多分 forklore に属すものと思われるが, 特に **命題** 1 は集合論的トポロジーの方での連鎖条件の研究者が見落としていることが多いように見受けられるので, 一度書き出しておく価値があるのではないかと思う.

命題 2 から先では, forcing に関するごく初等的な知識を仮定している. これに関しては [5]

Based on . . .

The talk is based on two preprints:

- PBN, Damian Sobota, *On sequences of homomorphisms into measure algebras and the Efimov Problem* (arxiv)
- PBN, Katarzyna Cegiełka, *On measures induced by forcing names for ultrafilters*

Names for ultrafilters

Fix a Boolean algebra $\mathbb{A}$ and a forcing $\mathbb{P}$.

Names for ultrafilters

Fix a Boolean algebra $\mathbb{A}$ and a forcing $\mathbb{P}$.

Let G be a $\mathbb{P}$-generic over V. Then, in $V[G]$, we may consider $\mathbb{A}$ and ultrafilters on $\mathbb{A}$ (*old* and *new*).

Names for ultrafilters

Fix a Boolean algebra $\mathbb{A}$ and a forcing $\mathbb{P}$.

Let G be a $\mathbb{P}$-generic over V. Then, in $V[G]$, we may consider $\mathbb{A}$ and ultrafilters on $\mathbb{A}$ (*old* and *new*).

Applications:

- reals are ultrafilters on the Cantor algebra,

Names for ultrafilters

Fix a Boolean algebra $\mathbb{A}$ and a forcing $\mathbb{P}$.

Let G be a $\mathbb{P}$-generic over V. Then, in $V[G]$, we may consider $\mathbb{A}$ and ultrafilters on $\mathbb{A}$ (*old* and *new*).

Applications:

- reals are ultrafilters on the Cantor algebra,
- Stone spaces of *old* Boolean algebras may provide interesting examples of topological spaces.

A name induces a homomorphism

Fix a Boolean algebra $\mathbb{A}$ and a forcing $\mathbb{P}$.

A name induces a homomorphism

Fix a Boolean algebra $\mathbb{A}$ and a forcing $\mathbb{P}$.

Let $\dot{u}$ be a $\mathbb{P}$-name for an ultrafilter on $\mathbb{A}$.

A name induces a homomorphism

Fix a Boolean algebra $\mathbb{A}$ and a forcing $\mathbb{P}$.

Let $\dot{u}$ be a $\mathbb{P}$-name for an ultrafilter on $\mathbb{A}$.
Consider $\varphi\colon \mathbb{A} \to \mathbb{P}$ defined by

$$\varphi(A) = \|A \in \dot{u}\|.$$

Then φ is a Boolean homomorphism.

A homomorphism induces a name

Fix a Boolean algebra $\mathbb{A}$ and a forcing $\mathbb{P}$.

A homomorphism induces a name

Fix a Boolean algebra $\mathbb{A}$ and a forcing $\mathbb{P}$.

Let $\varphi\colon \mathbb{A} \to \mathbb{P}$ be a Boolean homomorphism.

A homomorphism induces a name

Fix a Boolean algebra $\mathbb{A}$ and a forcing $\mathbb{P}$.

Let $\varphi\colon \mathbb{A} \to \mathbb{P}$ be a Boolean homomorphism.
Consider a $\mathbb{P}$-name $\dot{u}$ defined by

$$\dot{\varphi} = \{\langle A, \varphi(A)\rangle \colon A \in \mathbb{A}\}.$$

Then $\dot{\varphi}$ is a name for an ultrafilter on $\mathbb{A}$.

Names of ultrafilters and Boolean homomorphisms

Fix a Boolean algebra $\mathbb{A}$ and a forcing $\mathbb{P}$.

Names of ultrafilters and Boolean homomorphisms

Fix a Boolean algebra $\mathbb{A}$ and a forcing $\mathbb{P}$.

Proposition

For every $\mathbb{P}$-name $\dot{u}$ for an ultrafilter on $\mathbb{A}$ there is a Boolean homomorphism $\varphi\colon \mathbb{A} \to \mathbb{P}$ such that

$$1 \Vdash \dot{u} = \dot{\varphi}.$$

Names of ultrafilters and Boolean homomorphisms

Fix a Boolean algebra $\mathbb{A}$ and a forcing $\mathbb{P}$.

Proposition

For every $\mathbb{P}$-name $\dot{u}$ for an ultrafilter on $\mathbb{A}$ there is a Boolean homomorphism $\varphi\colon \mathbb{A} \to \mathbb{P}$ such that

$$1 \Vdash \dot{u} = \dot{\varphi}.$$

Remark. Ultrafilters = homomorphisms to $\{0, 1\}$.

Measure algebras.

Definition

For a cardinal number κ define *the measure algebra of type* κ by

$$\mathbb{M}_\kappa = \mathrm{Bor}(\{0,1\}^\kappa)_{/\lambda_\kappa = 0}.$$

Measure algebras.

Definition

For a cardinal number κ define *the measure algebra of type* κ by

$$\mathbb{M}_\kappa = \mathrm{Bor}(\{0,1\}^\kappa)_{/\lambda_\kappa = 0}.$$

Remarks:

Measure algebras.

Definition

For a cardinal number κ define *the measure algebra of type* κ by

$$\mathbb{M}_\kappa = \mathrm{Bor}(\{0,1\}^\kappa)_{/\lambda_\kappa = 0}.$$

Remarks:

- $\mathbb{M}_\kappa$ supports the standard Haar measure λ_κ,

Measure algebras.

Definition

For a cardinal number κ define *the measure algebra of type* κ by

$$\mathbb{M}_\kappa = \mathrm{Bor}(\{0,1\}^\kappa)_{/\lambda_\kappa = 0}.$$

Remarks:

- $\mathbb{M}_\kappa$ supports the standard Haar measure λ_κ,
- $\mathbb{M}_1 = \{0,1\}$,

Measure algebras.

Definition

For a cardinal number κ define *the measure algebra of type* κ by

$$\mathbb{M}_\kappa = \mathrm{Bor}(\{0,1\}^\kappa)_{/\lambda_\kappa = 0}.$$

Remarks:

- $\mathbb{M}_\kappa$ supports the standard Haar measure λ_κ,
- $\mathbb{M}_1 = \{0,1\}$,
- $\mathbb{M}_\omega = \mathrm{Bor}[0,1]_{/\mathcal{N}}$,

Measure algebras.

Definition

For a cardinal number κ define *the measure algebra of type* κ by

$$\mathbb{M}_\kappa = \mathrm{Bor}(\{0,1\}^\kappa)_{/\lambda_\kappa=0}.$$

Remarks:

- $\mathbb{M}_\kappa$ supports the standard Haar measure λ_κ,
- $\mathbb{M}_1 = \{0,1\}$,
- $\mathbb{M}_\omega = \mathrm{Bor}[0,1]_{/\mathcal{N}}$,
- Forcing with $\mathbb{M}_\kappa$ = adding κ random reals (for $\kappa > \omega$).

Measures induced by homomorphisms.

Let $\mathbb{A}$ be a Boolean algebra and $\kappa \geq \omega$.

Measures induced by homomorphisms.

Let $\mathbb{A}$ be a Boolean algebra and $\kappa \geq \omega$.

Let $\varphi\colon \mathbb{A} \to \mathbb{M}_\kappa$ be a Boolean homomorphism. Then φ induces a measure μ on $\mathbb{A}$ defined by

Measures induced by homomorphisms.

Let $\mathbb{A}$ be a Boolean algebra and $\kappa \geq \omega$.

Let $\varphi\colon \mathbb{A} \to \mathbb{M}_\kappa$ be a Boolean homomorphism. Then φ induces a measure μ on $\mathbb{A}$ defined by

$$\mu(A) = \lambda_\kappa(\varphi(A)).$$

Measures induced by homomorphisms.

Let $\mathbb{A}$ be a Boolean algebra and $\kappa \geq \omega$.

Let $\varphi \colon \mathbb{A} \to \mathbb{M}_\kappa$ be a Boolean homomorphism. Then φ induces a measure μ on $\mathbb{A}$ defined by

$$\mu(A) = \lambda_\kappa(\varphi(A)).$$

Remark. By a "measure" we mean here a *finitely additive* measure.

Examples

Fix a Boolean algebra $\mathbb{A}$, and a measure algebra $\mathbb{M}$.

Examples

Fix a Boolean algebra $\mathbb{A}$, and a measure algebra $\mathbb{M}$.

Let $\dot{\varphi}$ be a $\mathbb{M}$-name for an ultrafilter on $\mathbb{A}$. Let μ be the measure induced by φ.

Examples

Fix a Boolean algebra $\mathbb{A}$, and a measure algebra $\mathbb{M}$.

Let $\dot{\varphi}$ be a $\mathbb{M}$-name for an ultrafilter on $\mathbb{A}$. Let μ be the measure induced by φ.

- Suppose that $\mu = \delta_u$ for some ultrafilter u on $\mathbb{A}$ (in V!). Then ...

Examples

Fix a Boolean algebra $\mathbb{A}$, and a measure algebra $\mathbb{M}$.

Let $\dot{\varphi}$ be a $\mathbb{M}$-name for an ultrafilter on $\mathbb{A}$. Let μ be the measure induced by φ.

- Suppose that $\mu = \delta_u$ for some ultrafilter u on $\mathbb{A}$ (in V!). Then ...

$$1 \Vdash \dot{\varphi} = \check{u}.$$

Examples

Fix a Boolean algebra $\mathbb{A}$, and a measure algebra $\mathbb{M}$.

Let $\dot{\varphi}$ be a $\mathbb{M}$-name for an ultrafilter on $\mathbb{A}$. Let μ be the measure induced by φ.

- Suppose that $\mu = \delta_u$ for some ultrafilter u on $\mathbb{A}$ (in V!). Then ...

$$1 \Vdash \dot{\varphi} = \check{u}.$$

- If μ is purely atomic, then

$$1 \Vdash \dot{\varphi} \in V.$$

Examples

Let $\mathbb{C}$ be Cantor algebra. Fix a measure algebra $\mathbb{M}$.

Examples

Let $\mathbb{C}$ be Cantor algebra. Fix a measure algebra $\mathbb{M}$.

Let $\dot{\varphi}$ be a $\mathbb{M}$-name for an ultrafilter on $\mathbb{C}$ (a real). Let μ be the measure induced by φ.

Examples

Let $\mathbb{C}$ be Cantor algebra. Fix a measure algebra $\mathbb{M}$.

Let $\dot{\varphi}$ be a $\mathbb{M}$-name for an ultrafilter on $\mathbb{C}$ (a real). Let μ be the measure induced by φ.

- if μ is the standard Lebesgue measure, then $\dot{\varphi}$ is a "random" real.

Examples

Let $\mathbb{C}$ be Cantor algebra. Fix a measure algebra $\mathbb{M}$.

Let $\dot{\varphi}$ be a $\mathbb{M}$-name for an ultrafilter on $\mathbb{C}$ (a real). Let μ be the measure induced by φ.

- if μ is the standard Lebesgue measure, then $\dot{\varphi}$ is a "random" real.
- if μ is non-atomic, then $\dot{\varphi}$ is a "new" real.

Application: "reals".

Kunen's theorem

Theorem (Kunen)

In the classical random model there are no well ordered chains of size ω_2 in $\mathcal{P}(\omega)/Fin$.

Kunen's theorem

Theorem (Kunen)

In the classical random model there are no well ordered chains of size ω_2 in $\mathcal{P}(\omega)/Fin$.

Proposition

Assume GCH. *Suppose $(\varphi_\alpha)_{\alpha<\omega_2}$ is a sequence of homomorphisms $\varphi_\alpha\colon \mathbb{C} \to \mathbb{M}_{\omega_2}$. Then there is an automorphisms $\Phi\colon \mathbb{M}_{\omega_2} \to \mathbb{M}_{\omega_2}$ such that $\Phi \circ \varphi_\alpha = \varphi_\beta$ and $\Phi \circ \varphi_\beta = \varphi_\alpha$.*

Kunen's theorem

Theorem (Kunen)

In the classical random model there are no well ordered chains of size ω_2 in $\mathcal{P}(\omega)/Fin$.

Kunen's theorem

Theorem (Kunen)

In the classical random model there are no well ordered chains of size ω_2 in $\mathcal{P}(\omega)/Fin$.

- Suppose the contrary. I.e.
  $1 \Vdash$ there is $(\dot{T}_\alpha)_{\alpha<\omega_2}$ strictly $\subseteq^*$ -increasing in $\mathcal{P}(\omega)$.

Kunen's theorem

Theorem (Kunen)

In the classical random model there are no well ordered chains of size ω_2 in $\mathcal{P}(\omega)/Fin$.

- Suppose the contrary. I.e.
  $1 \Vdash$ there is $(\dot{T}_\alpha)_{\alpha<\omega_2}$ strictly $\subseteq^*$ -increasing in $\mathcal{P}(\omega)$.
- We may assume that $\dot{T}_\alpha = \dot{\varphi}_\alpha$ for $\alpha < \omega_2$ and a homomorphism $\varphi_\alpha \colon \mathbb{C} \to \mathbb{M}_{\omega_2}$.

Kunen's theorem

Theorem (Kunen)

In the classical random model there are no well ordered chains of size ω_2 in $\mathcal{P}(\omega)/Fin$.

- Suppose the contrary. I.e.
  $1 \Vdash$ there is $(\dot{T}_\alpha)_{\alpha<\omega_2}$ strictly $\subseteq^*$ -increasing in $\mathcal{P}(\omega)$.
- We may assume that $\dot{T}_\alpha = \dot{\varphi}_\alpha$ for $\alpha < \omega_2$ and a homomorphism $\varphi_\alpha \colon \mathbb{C} \to \mathbb{M}_{\omega_2}$.
- Then we have an automorphism Φ such that
  $\Phi[\mathbb{M}_{\omega_2}] \Vdash \Phi \dot{\circ} \varphi_\alpha \subseteq^* \Phi \dot{\circ} \varphi_\beta$.

Kunen's theorem

Theorem (Kunen)

In the classical random model there are no well ordered chains of size ω_2 in $\mathcal{P}(\omega)/Fin$.

- Suppose the contrary. I.e.
  $1 \Vdash$ there is $(\dot{T}_\alpha)_{\alpha<\omega_2}$ strictly $\subseteq^*$ -increasing in $\mathcal{P}(\omega)$.
- We may assume that $\dot{T}_\alpha = \dot{\varphi}_\alpha$ for $\alpha < \omega_2$ and a homomorphism $\varphi_\alpha \colon \mathbb{C} \to \mathbb{M}_{\omega_2}$.
- Then we have an automorphism Φ such that $\Phi[\mathbb{M}_{\omega_2}] \Vdash \Phi \dot{\circ} \varphi_\alpha \subseteq^* \Phi \dot{\circ} \varphi_\beta$.
- But this means that

$$\mathbb{M}_{\omega_2} \Vdash \dot{\varphi}_\beta \subseteq^* \dot{\varphi}_\alpha.$$

Kunen's theorem

Proposition

Assume GCH*. Suppose* $(\varphi_\alpha)_{\alpha<\omega_2}$ *is a sequence of homomorphisms* $\varphi_\alpha \colon \mathbb{C} \to \mathbb{M}_{\omega_2}$*. Then there is an automorphisms* $\Phi \colon \mathbb{M}_{\omega_2} \to \mathbb{M}_{\omega_2}$ *such that* $\Phi \circ \varphi_\alpha = \varphi_\beta$ *and* $\Phi \circ \varphi_\beta = \varphi_\alpha$*.*

- WLOG: there is a measure μ on $\mathbb{C}$ such that $\mu = \lambda_{\omega_2} \circ \varphi_\alpha$ for each α.

Kunen's theorem

Proposition

Assume GCH. *Suppose* $(\varphi_\alpha)_{\alpha<\omega_2}$ *is a sequence of homomorphisms* $\varphi_\alpha\colon \mathbb{C} \to \mathbb{M}_{\omega_2}$. *Then there is an automorphisms* $\Phi\colon \mathbb{M}_{\omega_2} \to \mathbb{M}_{\omega_2}$ *such that* $\Phi \circ \varphi_\alpha = \varphi_\beta$ *and* $\Phi \circ \varphi_\beta = \varphi_\alpha$.

- WLOG: there is a measure μ on $\mathbb{C}$ such that $\mu = \lambda_{\omega_2} \circ \varphi_\alpha$ for each α.
- Fix an independent family $\{C_n\colon n \in \omega\} \subseteq \mathbb{C}$ generating $\mathbb{C}$.

Kunen's theorem

Proposition

Assume GCH. *Suppose* $(\varphi_\alpha)_{\alpha<\omega_2}$ *is a sequence of homomorphisms* $\varphi_\alpha\colon \mathbb{C} \to \mathbb{M}_{\omega_2}$. *Then there is an automorphisms* $\Phi\colon \mathbb{M}_{\omega_2} \to \mathbb{M}_{\omega_2}$ *such that* $\Phi \circ \varphi_\alpha = \varphi_\beta$ *and* $\Phi \circ \varphi_\beta = \varphi_\alpha$.

- WLOG: there is a measure μ on $\mathbb{C}$ such that $\mu = \lambda_{\omega_2} \circ \varphi_\alpha$ for each α.
- Fix an independent family $\{C_n\colon n \in \omega\} \subseteq \mathbb{C}$ generating $\mathbb{C}$.
- We say that $A \in \mathbb{C}$ is *a chunk* if it is of the form

$$A = (C_{i_0} \wedge \cdots \wedge C_{i_k}) \wedge (C^c_{j_0} \wedge \cdots \wedge C^c_{j_l})$$

Kunen's theorem

Proposition

Assume GCH. *Suppose* $(\varphi_\alpha)_{\alpha<\omega_2}$ *is a sequence of homomorphisms* $\varphi_\alpha\colon \mathbb{C} \to \mathbb{M}_{\omega_2}$. *Then there is an automorphisms* $\Phi\colon \mathbb{M}_{\omega_2} \to \mathbb{M}_{\omega_2}$ *such that* $\Phi \circ \varphi_\alpha = \varphi_\beta$ *and* $\Phi \circ \varphi_\beta = \varphi_\alpha$.

- WLOG: there is a measure μ on $\mathbb{C}$ such that $\mu = \lambda_{\omega_2} \circ \varphi_\alpha$ for each α.
- Fix an independent family $\{C_n\colon n \in \omega\} \subseteq \mathbb{C}$ generating $\mathbb{C}$.
- We say that $A \in \mathbb{C}$ is *a chunk* if it is of the form

$$A = (C_{i_0} \wedge \cdots \wedge C_{i_k}) \wedge (C^c_{j_0} \wedge \cdots \wedge C^c_{j_l})$$

- We say that α and β are *symmetric* if for each chunks A, B

$$\varphi_\alpha(A) \wedge \varphi_\beta(B) = 0 \iff \varphi_\beta(A) \wedge \varphi_\alpha(B) = 0$$

Kunen's theorem

Claim. There are $\alpha < \beta < \omega_2$ which are symmetric, i.e.

$$\varphi_\alpha(A) \wedge \varphi_\beta(B) = 0 \iff \varphi_\beta(A) \wedge \varphi_\alpha(B) = 0$$

for every chunks A and B.

Kunen's theorem

Claim. There are $\alpha < \beta < \omega_2$ which are symmetric, i.e.

$$\varphi_\alpha(A) \wedge \varphi_\beta(B) = 0 \iff \varphi_\beta(A) \wedge \varphi_\alpha(B) = 0$$

for every chunks A and B.

- If not, then there is a coloring $c\colon [\omega_2]^2 \to \mathbb{C} \times \mathbb{C}$ assigning "witnesses".

Kunen's theorem

Claim. There are $\alpha < \beta < \omega_2$ which are symmetric, i.e.

$$\varphi_\alpha(A) \wedge \varphi_\beta(B) = 0 \iff \varphi_\beta(A) \wedge \varphi_\alpha(B) = 0$$

for every chunks A and B.

- If not, then there is a coloring $c \colon [\omega_2]^2 \to \mathbb{C} \times \mathbb{C}$ assigning "witnesses".
- GCH + Erdös-Rado implies that there is an uncountable monochromatic Λ (WLOG $= \omega_1$) with color $\langle A, B \rangle$.

Kunen's theorem

Claim. There are $\alpha < \beta < \omega_2$ which are symmetric, i.e.

$$\varphi_\alpha(A) \wedge \varphi_\beta(B) = 0 \iff \varphi_\beta(A) \wedge \varphi_\alpha(B) = 0$$

for every chunks A and B.

- If not, then there is a coloring $c\colon [\omega_2]^2 \to \mathbb{C} \times \mathbb{C}$ assigning "witnesses".
- GCH + Erdös-Rado implies that there is an uncountable monochromatic Λ (WLOG $= \omega_1$) with color $\langle A, B \rangle$.
- "Then" for each $\alpha < \beta < \omega_1$ we have $\varphi_\alpha(A) \wedge \varphi_\beta(B) = 0$ and $\varphi_\beta(A) \wedge \varphi_\alpha(B) \neq 0$.

Kunen's theorem

- For each $\alpha < \beta < \omega_1$ we have $\varphi_\alpha(A) \wedge \varphi_\beta(B) = 0$ and $\varphi_\beta(A) \wedge \varphi_\alpha(B) \neq 0$.

Kunen's theorem

- For each $\alpha < \beta < \omega_1$ we have $\varphi_\alpha(A) \wedge \varphi_\beta(B) = 0$ and $\varphi_\beta(A) \wedge \varphi_\alpha(B) \neq 0$.
- Let $D = \bigvee_{\alpha<\omega_1} \varphi_\alpha(A)$.

Kunen's theorem

- For each $\alpha < \beta < \omega_1$ we have $\varphi_\alpha(A) \wedge \varphi_\beta(B) = 0$ and $\varphi_\beta(A) \wedge \varphi_\alpha(B) \neq 0$.
- Let $D = \bigvee_{\alpha<\omega_1} \varphi_\alpha(A)$.
- By ccc there is $\gamma < \omega_1$ such that $D = \bigvee_{\alpha<\gamma} \varphi_\alpha(A)$.

Kunen's theorem

- For each $\alpha < \beta < \omega_1$ we have $\varphi_\alpha(A) \wedge \varphi_\beta(B) = 0$ and $\varphi_\beta(A) \wedge \varphi_\alpha(B) \neq 0$.
- Let $D = \bigvee_{\alpha<\omega_1} \varphi_\alpha(A)$.
- By ccc there is $\gamma < \omega_1$ such that $D = \bigvee_{\alpha<\gamma} \varphi_\alpha(A)$.
- Then $D \wedge \varphi_\gamma(B) = 0$.

Kunen's theorem

- For each $\alpha < \beta < \omega_1$ we have $\varphi_\alpha(A) \wedge \varphi_\beta(B) = 0$ and $\varphi_\beta(A) \wedge \varphi_\alpha(B) \neq 0$.
- Let $D = \bigvee_{\alpha<\omega_1} \varphi_\alpha(A)$.
- By ccc there is $\gamma < \omega_1$ such that $D = \bigvee_{\alpha<\gamma} \varphi_\alpha(A)$.
- Then $D \wedge \varphi_\gamma(B) = 0$.
- But $\varphi_{\gamma+1}(A) \wedge \varphi_\gamma(B) \neq 0$ and $\varphi_{\gamma+1}(A) \leq D$. A contradiction.

Kunen's theorem

Claim. There are α and $\beta < \omega_2$ which are symmetric, i.e.

$$\varphi_\alpha(A) \wedge \varphi_\beta(B) = 0 \iff \varphi_\beta(A) \wedge \varphi_\alpha(B) = 0$$

for each chunks A and B.

Kunen's theorem

Claim. There are α and $\beta < \omega_2$ which are symmetric, i.e.

$$\varphi_\alpha(A) \wedge \varphi_\beta(B) = 0 \iff \varphi_\beta(A) \wedge \varphi_\alpha(B) = 0$$

for each chunks A and B.

So, by Sikorski's Extension Lemma, there is an automorphism $\Phi\colon \mathbb{M} \to \mathbb{M}$ such that

$$\Phi \circ \varphi_\alpha = \varphi_\beta \text{ and } \Phi \circ \varphi_\beta = \varphi_\alpha.$$

Kunen's theorem

Claim. There are α and $\beta < \omega_2$ which are symmetric, i.e.

$$\varphi_\alpha(A) \wedge \varphi_\beta(B) = 0 \iff \varphi_\beta(A) \wedge \varphi_\alpha(B) = 0$$

for each chunks A and B.

So, by Sikorski's Extension Lemma, there is an automorphism $\Phi\colon \mathbb{M} \to \mathbb{M}$ such that

$$\Phi \circ \varphi_\alpha = \varphi_\beta \text{ and } \Phi \circ \varphi_\beta = \varphi_\alpha.$$

and we are done.

Kunen's theorem

Theorem (Kunen)

In the classical random model there are no well ordered chains of size ω_2 in $\mathcal{P}(\omega)/Fin$.

Application: peculiar topological spaces

Efimov problem

Theorem (Dow, Fremlin)

In the classical random model there is a Efimov space, i.e. an infinite compact space without a nontrivial convergent sequences and without a copy of $\beta\omega$.

Frechet-Nikodym metric

Definition

If $\mathbb{M}$ is a measure algebra, then $d_\lambda \colon \mathbb{M} \to [0, \infty)$ defined by

$$d_\lambda(A, B) = \mu(A \triangle B)$$

is a metric (called *Frechet-Nikodym metric*).

Frechet-Nikodym metric

Definition

If $\mathbb{M}$ is a measure algebra, then $d_\lambda \colon \mathbb{M} \to [0, \infty)$ defined by

$$d_\lambda(A, B) = \mu(A \triangle B)$$

is a metric (called *Frechet-Nikodym metric*).

Definition

We say that a sequence of homomorphisms $\varphi_n \colon \mathbb{A} \to \mathbb{M}$ converges *metrically pointwise* to a homomorphism $\varphi \colon \mathbb{A} \to \mathbb{M}$ if

$$d_\lambda(\varphi_n(A), \varphi(A)) \to 0.$$

Frechet-Nikodym metric

Definition

If $\mathbb{M}$ is a measure algebra, then $d_\lambda \colon \mathbb{M} \to [0,\infty)$ defined by

$$d_\lambda(A,B) = \mu(A \triangle B)$$

is a metric (called *Frechet-Nikodym metric*).

Definition

We say that a sequence of homomorphisms $\varphi_n \colon \mathbb{A} \to \mathbb{M}$ converges *metrically pointwise* to a homomorphism $\varphi \colon \mathbb{A} \to \mathbb{M}$ if

$$d_\lambda(\varphi_n(A), \varphi(A)) \to 0.$$

Remark. Stone topology = pointwise convergence topology.

Pointwise convergence and convergence in the extension

Proposition (PBN, Sobota)

Suppose that

$$\mathbb{M} \Vdash (\dot{\varphi}_n) \text{ converges to } \dot{\varphi}.$$

Then (φ_n) *converges metrically pointwise to* φ.

Pointwise convergence and convergence in the extension

Proposition (PBN, Sobota)

Suppose that

$$\mathbb{M} \Vdash (\dot{\varphi}_n) \text{ converges to } \dot{\varphi}.$$

Then (φ_n) converges metrically pointwise to φ. Consequently, $\lambda \circ \varphi_n$ converges weakly* to $\lambda \circ \varphi$.

Pointwise algebraic convergence and convergence in the extension

Definition

We say that a sequence (A_n) in a Boolean algebra *converges algebraically* to A if

$$\bigvee \bigwedge A_n = \bigwedge \bigvee A_n = A.$$

Pointwise algebraic convergence and convergence in the extension

Definition

We say that a sequence (A_n) in a Boolean algebra *converges algebraically* to A if

$$\bigvee \bigwedge A_n = \bigwedge \bigvee A_n = A.$$

Proposition (PBN, Sobota)

Let $\mathbb{M}$ be a measure algebra. Then

$$\mathbb{M} \Vdash (\dot{\varphi}_n) \text{ converges to } \dot{\varphi}$$

if and only if

(φ_n) converges to φ pointwise algebraically.

Uniform convergence

Definition

We say that a sequence of homomorphisms $\varphi_n \colon \mathbb{A} \to \mathbb{M}$ converges *uniformly* to a homomorphism $\varphi \colon \mathbb{A} \to \mathbb{M}$ if

$$\forall \varepsilon > 0 \ \exists N \ \forall n > N \ \forall A \in \mathbb{A} \ d_\lambda(\varphi_n(A), \varphi(A)) < \varepsilon.$$

Uniform convergence

Definition

We say that a sequence of homomorphisms $\varphi_n\colon \mathbb{A} \to \mathbb{M}$ converges *uniformly* to a homomorphism $\varphi\colon \mathbb{A} \to \mathbb{M}$ if

$$\forall \varepsilon > 0\ \exists N\ \forall n > N\ \forall A \in \mathbb{A}\ d_\lambda(\varphi_n(A), \varphi(A)) < \varepsilon.$$

Theorem (PBN, Sobota, 2020)

Let $\mathbb{M}$ be a measure algebra. If $\mathbb{M} \Vdash (\dot\varphi_n)$ converges trivially to $\dot\varphi$, then (φ_n) converges to φ uniformly.

Uniform convergence

Definition

We say that a sequence of homomorphisms $\varphi_n \colon \mathbb{A} \to \mathbb{M}$ converges *uniformly* to a homomorphism $\varphi \colon \mathbb{A} \to \mathbb{M}$ if

$$\forall \varepsilon > 0 \; \exists N \; \forall n > N \; \forall A \in \mathbb{A} \; d_\lambda(\varphi_n(A), \varphi(A)) < \varepsilon.$$

Theorem (PBN, Sobota, 2020)

Let $\mathbb{M}$ be a measure algebra. If $\mathbb{M} \Vdash (\dot{\varphi}_n)$ converges trivially to $\dot{\varphi}$, then (φ_n) converges to φ uniformly.

If $\mathbb{M} \Vdash (\dot{\varphi}_n)$ converges non-trivially to $\dot{\varphi}$, then (φ_n) does not converge to φ uniformly.

Names

Assume $1 \Vdash \dot{\varphi} \neq \dot{\psi}$.

Names

Assume $1 \Vdash \dot{\varphi} \neq \dot{\psi}$.
Then

$$1 \Vdash \exists A \in \mathbb{A}\ A \in \dot{\varphi} \setminus \dot{\psi}.$$

Names

Assume $1 \Vdash \dot{\varphi} \neq \dot{\psi}$.
Then

$$1 \Vdash \exists A \in \mathbb{A} \ A \in \dot{\varphi} \setminus \dot{\psi}.$$

There is an antichain (p_n) in $\mathbb{M}$ and a sequence (A_n) of elements of $\mathbb{A}$ such that

$$p_n \Vdash A_n \in \dot{\varphi} \setminus \dot{\psi}.$$

Names

Assume $1 \Vdash \dot{\varphi} \neq \dot{\psi}$.
Then

$$1 \Vdash \exists A \in \mathbb{A} \ A \in \dot{\varphi} \setminus \dot{\psi}.$$

There is an antichain (p_n) in $\mathbb{M}$ and a sequence (A_n) of elements of $\mathbb{A}$ such that

$$p_n \Vdash A_n \in \dot{\varphi} \setminus \dot{\psi}.$$

Can we say anything about $\lambda(p_n)$?

A fence exercise

Fence exercise

Shimamura and Komako are painting a picket fence between their properties. The fence consists of n many rails. Each rail has two sides—Shimamura's and Komako's—and those sides have to be painted in such a way that Shimamura's side has different colour than Komako's.

Fence exercise

Shimamura and Komako are painting a picket fence between their properties. The fence consists of n many rails. Each rail has two sides—Shimamura's and Komako's—and those sides have to be painted in such a way that Shimamura's side has different colour than Komako's.

No matter how many colours they use, there is always a set B of at least $n/4$ many rails with the following property: the set of Shimamura's colours used in B is disjoint with the set of Komako's colours used in B.

Fence exercise

Shimamura and Komako are painting a picket fence between their properties. The fence consists of n many rails. Each rail has two sides—Shimamura's and Komako's—and those sides have to be painted in such a way that Shimamura's side has different colour than Komako's.

No matter how many colours they use, there is always a set B of at least $n/4$ many rails with the following property: the set of Shimamura's colours used in B is disjoint with the set of Komako's colours used in B.

[I will present a proof by Dominik Gdesz]

The 1/4 theorem

Theorem (PBN, Sobota)

Let $\mathbb{M}$ *be a measure algebra. If*

$$\mathbb{M} \Vdash \exists A \in \mathbb{A}\ A \in \dot{\varphi} \setminus \dot{\psi},$$

then there is $p \in \mathbb{M}$, $A \in \mathbb{A}$ *such that* $\lambda(p) > 1/4$ *and*

$$p \Vdash A \in \dot{\varphi} \setminus \dot{\psi}.$$

The 1/4 theorem

Theorem (PBN, Sobota)

Let $\mathbb{M}$ *be a measure algebra. If*

$$\mathbb{M} \Vdash \exists A \in \mathbb{A} \ A \in \dot{\varphi} \setminus \dot{\psi},$$

then there is $p \in \mathbb{M}$, $A \in \mathbb{A}$ *such that* $\lambda(p) > 1/4$ *and*

$$p \Vdash A \in \dot{\varphi} \setminus \dot{\psi}.$$

The crux of the proof: fence exercise.

The 1/4 theorem

Theorem (PBN, Sobota)

Let $\mathbb{M}$ *be a measure algebra. If*

$$\mathbb{M} \Vdash \exists A \in \mathbb{A}\ A \in \dot{\varphi} \setminus \dot{\psi},$$

then there is $p \in \mathbb{M}$, $A \in \mathbb{A}$ *such that* $\lambda(p) > 1/4$ *and*

$$p \Vdash A \in \dot{\varphi} \setminus \dot{\psi}.$$

The crux of the proof: fence exercise.

Efimov

Theorem (PBN, Sobota)

Let $\mathbb{A}$ be a Boolean algebra, and let $\mathbb{M}$ be a measure algebra. Then the following are equivalent:

- $\mathbb{M} \Vdash \mathrm{St}(\mathbb{A})$ *does not have non-trivial convergent sequences;*

Efimov

Theorem (PBN, Sobota)

Let $\mathbb{A}$ be a Boolean algebra, and let $\mathbb{M}$ be a measure algebra. Then the following are equivalent:

- $\mathbb{M} \Vdash \mathrm{St}(\mathbb{A})$ *does not have non-trivial convergent sequences;*
- *every pointwise algebraically convergent sequence of homomorphisms from $\mathbb{A}$ into $\mathbb{M}_\kappa$ converges uniformly.*

Efimov

Theorem (PBN, Sobota)

Let $\mathbb{A}$ be a Boolean algebra, and let $\mathbb{M}$ be a measure algebra. Then the following are equivalent:

- $\mathbb{M} \Vdash \mathrm{St}(\mathbb{A})$ *does not have non-trivial convergent sequences;*
- *every pointwise algebraically convergent sequence of homomorphisms from $\mathbb{A}$ into $\mathbb{M}_\kappa$ converges uniformly.*

Corollary

Every pointwise algebraically convergent sequence of homomorphisms from $\mathcal{P}(\omega)/Fin$ into $\mathbb{M}_\kappa$ converges uniformly.

Efimov

Corollary

Every pointwise algebraically convergent sequence of homomorphisms from $\mathcal{P}(\omega)/Fin$ into $\mathbb{M}_\kappa$ converges uniformly.

Efimov

Corollary

Every pointwise algebraically convergent sequence of homomorphisms from $\mathcal{P}(\omega)/Fin$ into $\mathbb{M}_\kappa$ converges uniformly.

Theorem (PBN, Cegiełka)

For each $\kappa \geq \omega$ and a Boolean algebra $\mathbb{A}$ there is a sequence of homomorphisms from $\mathbb{A}$ into $\mathbb{M}_\kappa$ which converges pointwise metrically but not uniformly.

Thank you