Superstrong and Huge Reflection

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Structural Reflection

SR: (*Structural Reflection*) *For every definable, in the first-order language of set theory (possibly with parameters), class* $\mathcal{C}$ *of relational structures of the same type there exists an ordinal* α *that reflects* $\mathcal{C}$*, i.e., for every* A *in* $\mathcal{C}$ *there exists* B *in* $\mathcal{C} \cap V_\alpha$ *and an elementary embedding from* B *into* A*.*

The $\mathcal{SR}$ principle may be properly formulated in the first-order language of set theory as an axiom schema, to wit, for each natural number n, and set P, let

$\Sigma_n(P)$-$\mathcal{SR}$: (Σ_n-*Structural Reflection*) *For every Σ_n-definable, with parameters in P, class $\mathcal{C}$ of relational structures of the same type there is an ordinal α that reflects $\mathcal{C}$.*

$\Pi_n(P)$-$\mathcal{SR}$ may be defined analogously.

SR from Supercompact to Vopěnka's Principle

Table 1[1]

Complexity	SR
Σ_1	ZFC
Π_1, Σ_2	Supercompact
Π_2, Σ_3	Extendible
Π_3, Σ_4	$C^{(2)}$-Extendible
$\vdots$	$\vdots$
Π_n, Σ_{n+1}	$C^{(n-1)}$-Extendible
$\vdots$	$\vdots$
Π_n, all n	VP

[1] J. Bagaria. $C^{(n)}$-cardinals. *Archive for Math. Logic*, 51:213–240, 2012. J. Bagaria, C. Casacuberta, A. R. D. Mathias, and J. Rosický. Definable orthogonality classes in accessible categories are small. *Journal of the European Mathematical Society*, 17(3):549–589, 2015.

Product $\mathcal{SR}$

PSR: (*Product Structural Reflection*) *For every (definable) class of relational structures* $\mathcal{C}$ *of the same type,* τ, *there exists an ordinal* α *that product-reflects* $\mathcal{C}$, *i.e., for every* $\mathcal{A}$ *in* $\mathcal{C}$ *there exists a set* $\mathcal{S}$ *of structures of type* τ *(although not necessarily in* $\mathcal{C}$*) with* $\mathcal{A} \in \mathcal{S}$ *and an elementary embedding* $j : \prod(\mathcal{C} \cap V_\alpha) \to \prod \mathcal{S}$.

Product $\mathcal{SR}$

For Γ a definability class (i.e., Σ_n or Π_n, some $n > 0$), let:

Γ-PSR: (Γ-*Product Structural Reflection*) *There exists a (proper class of) cardinal(s)* κ *that product-reflect all* Γ*-definable, with parameters in* V_κ*, class* $\mathcal{C}$ *of structures of the same type.*

Remark

We may, equivalently, restrict to definable classes of natural structures, namely structures of the form $\langle V_\alpha, \in, A\rangle$*, where* $A \subseteq V_\alpha$.

SR from Strong to ORD is Woodin

Table 2[2]

Complexity	PSR
Σ_1	ZFC
Π_1, Σ_2	Strong
Π_2, Σ_3	Π_2-Strong
Π_3, Σ_4	Π_3-Strong
$\vdots$	$\vdots$
Π_n, Σ_{n+1}	Π_n-Strong
$\vdots$	$\vdots$
Π_n, all n	ORD is Woodin

[2] J. Bagaria and T. Wilson. The Weak Vopěnka Principle for definable classes of structures. *To appear*, 2020.

Strong cardinals

Recall that a cardinal κ is strong if for every ordinal λ there exists an elementary embedding $j : V \to M$, with M transitive, $\text{crit} j = \kappa$, and $V_\lambda \subseteq M$.

More generally, we define:

Definition

A cardinal κ is Σ_n-*strong* if for every Σ_n-definable (with parameters in V_κ) class A, for every ordinal λ there exists an elementary embedding $j : V \to M$, with M transitive, $\text{crit}(j) = \kappa$, $V_\lambda \subseteq M$, and $A \cap V_\lambda \subseteq j(A)$.

Every strong cardinal is Σ_2-strong.
Also, if $\lambda \in C^{(n+1)}$, then a cardinal is λ-Π_n-strong iff it is λ-Σ_{n+1}-strong. So, Π_n-strong $\equiv$ Σ_{n+1}-strong.

ORD is Woodin

Definition ([3])

ORD *is Woodin* if for every definable $A \subseteq V$ there exists some α which is A-strong, i.e., for every γ there is an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \alpha$, $\gamma < j(\alpha)$, $V_\gamma \subseteq M$, and $A \cap V_\gamma = j(A) \cap V_\gamma$.

Note that if δ is a Woodin cardinal, then V_δ satisfies ORD is Woodin.

[3] J. Bagaria and T. Wilson. The Weak Vopěnka Principle for definable classes of structures. *To appear*, 2020.

SR from Strong to ORD is Woodin

Table 2[4]

Complexity	PSR
Σ_1	ZFC
Π_1, Σ_2	Strong
Π_2, Σ_3	Π_2-Strong
Π_3, Σ_4	Π_3-Strong
$\vdots$	$\vdots$
Π_n, Σ_{n+1}	Π_n-Strong
$\vdots$	$\vdots$
Π_n, all n	ORD is Woodin

[4] J. Bagaria and Trevor Wilson. The Weak Vopěnka Principle for definable classes of structures. *To appear*, 2020.

PSR redefined

If κ is a Π_n-strong cardinal, then for every Π_n-definable (with parameters in V_κ) class $\mathcal{C}$ of relational structures of the same type τ, and for every β, there exists a set $\mathcal{S}$ of structures of type τ (although possibly not in $\mathcal{C}$) that contains $\mathcal{C} \cap V_\beta$ and there exists an elementary embedding $h : \prod(\mathcal{C} \cap V_\kappa) \to \prod \mathcal{S}$ with the following additional properties:

1. Faithful: For every $f \in \prod(\mathcal{C} \cap V_\kappa)$, $h(f) \restriction (\mathcal{C} \cap V_\kappa) = f$.
2. $\subseteq$-chain-preserving: If $f \in \prod(\mathcal{C} \cap V_\kappa)$ is so that $f(\mathcal{A}) \subseteq f(\mathcal{A}')$ whenever $\mathcal{A} \subseteq \mathcal{A}'$, then so is $h(f)$.

PSR redefined

Thus, the following is an equivalent reformulation of Γ-PSR:

Γ-PSR: (Γ-*Product Structural Reflection. Second version*) *There exist a (proper class of) cardinal(s)* κ *that product-reflect all* Γ*-definable, with parameters in* V_κ*, proper class* $\mathcal{C}$ *of (natural) structures. I.e., for every* β *there exists a set* S *of (natural) structures of the same type that contains* $\mathcal{C} \cap V_\beta$ *and a faithful and* $\subseteq$*-chain-preserving elementary embedding* $h : \prod(\mathcal{C} \cap V_\kappa) \to \prod S$.

PSR redefined

Theorem

There is a Π_1-definable, without parameters, class $\mathcal{C}$ of natural structures such that if a cardinal κ product-reflects $\mathcal{C}$, then κ is a strong cardinal.

Let $\mathcal{C}$ be the Π_1-definable class of all structures

$$\mathcal{A}_\alpha := \langle V_{\lambda_\alpha}, \in, \alpha \rangle$$

where α is the α-th element of $C^{(1)}$ and λ_α is the least cardinal in $C^{(1)}$ greater than α.

Theorem

A cardinal κ is Π_n-strong if and only it it witnesses Π_n-PSR.

Strong Product Structural reflection

Consider now the following, arguably more natural, strengthening of PSR:

SPSR: (*Strong Product Structural Reflection*) *For every (definable) class of relational structures* $\mathcal{C}$ *of the same type,* τ*, there exists an ordinal* α *that strongly product-reflects* $\mathcal{C}$*, i.e., for every* $\mathcal{A}$ *in* $\mathcal{C}$ *there exists an ordinal* β *with* $\mathcal{A} \in V_\beta$ *and an elementary embedding* $j : \prod(\mathcal{C} \cap V_\alpha) \to \prod(\mathcal{C} \cap V_\beta)$.

Strong Product Structural reflection

We may formally define SPSR for classes of structures that are definable in the first-order language of set theory as an axiom schema:

Γ-SPSR: (Γ-*Strong Product Structural Reflection*) *There exists a (proper class of) cardinal(s)* κ *that strongly product-reflect all* Γ*-definable, with parameters in* V_κ*, class* $\mathcal{C}$ *of natural structures.*

Superstrong and globally superstrong cardinals

Definition ([5])

A cardinal κ is *superstrong above* λ, for some $\lambda \geqslant \kappa$, if there exists an elementary embedding $j : V \to M$, with M transitive, $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $V_{j(\kappa)} \subseteq M$.

A cardinal κ is *globally superstrong* if it is superstrong above λ, for every $\lambda \geqslant \kappa$.

A cardinal κ is $C^{(n)}$*-superstrong above* λ, for some $\lambda \geqslant \kappa$, if there exists an elementary embedding $j : V \to M$, with M transitive, $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $V_{j(\kappa)} \subseteq M$, and $j(\kappa) \in C^{(n)}$.

A cardinal κ is $C^{(n)}$*-globally superstrong* if it is $C^{(n)}$-superstrong above λ, for every $\lambda \geqslant \kappa$.

[5] J. Cai and K. Tsaprounis. On strengthenings of superstrong cardinals. *New York Journal of Mathematics*, 25:174–194, 2019.

Superstrong and globally superstrong cardinals

If κ is $C^{(n)}$-gobally superstrong, then there are many $C^{(n)}$-superstrong cardinals below κ.

If κ is $\kappa + 1$-extendible, then V_κ satisfies that there is a proper class of $C^{(n)}$-globally superstrong cardinals, for every n.

Moreover, if κ is $C^{(n)}$-extendible, then there are many $C^{(n)}$-globally superstrong cardinals below κ.

Superstrong and globally superstrong cardinals

Proposition

If κ is $C^{(n)}$-globally superstrong, then it witnesses Π_n-SPSR.

The proof of the last proposition shows that the witnessing elementary embeddings are faithful and $\subseteq$-chain preserving. Thus, we reformulate the SPSR schema as follows:

Γ-SPSR: (Π_n*-Strong Product Structural Reflection. Second version*) *There exist a (proper class of) cardinal(s) κ that strongly-product-reflect all Γ-definable, with parameters in V_κ, proper classes $\mathcal{C}$ of natural structures, i.e., for every $\mathcal{A} \in \mathcal{C}$ there exists an ordinal β with $\mathcal{A} \in V_\beta$ and a faithful and $\subseteq$-chain-preserving elementary embedding $h : \prod(\mathcal{C} \cap V_\kappa) \to \prod(\mathcal{C} \cap V_\beta)$.*

Superstrong and globally superstrong cardinals

Similarly as in the case of strong cardinals, we can now prove the following:

Theorem

There is a Π_n-definable, without parameters, class $\mathcal{C}$ of natural structures such that if a cardinal κ strongly-product-reflects $\mathcal{C}$, then κ is a $C^{(n)}$-globally superstrong cardinal.

SPSR and globally superstrong cardinals

Sketch of proof.

Let $\mathcal{C}$ be the Π_n-definable class of all structures

$$\mathcal{A}_\alpha := \langle V_{\lambda_\alpha}, \in, \alpha \rangle$$

where α has uncountable cofinality and is the α-th element of $C^{(n)}$, and λ_α is the least cardinal in $C^{(n)}$ greater than α.
Let κ witness *SPSR* for $\mathcal{C}$. Let $I := \{\alpha : \mathcal{A}_\alpha \in V_\kappa\}$. Since $\kappa \in C^{(n+1)}$, $\sup(I) = \kappa$.
Pick any ordinal $\lambda \geqslant \kappa$. We shall show that κ is λ-$C^{(n)}$-superstrong.
Let $\mathcal{A}_\beta$ in $\mathcal{C}$ be such that $\lambda < \beta$. Let κ' be such that $\mathcal{A}_\beta \in V_{\kappa'}$ and there is a faithful $\subseteq$-chain-preserving elementary embedding

$$j : \prod(\mathcal{C} \cap V_\kappa) \to \prod(\mathcal{C} \cap V_{\kappa'}).$$

□

Continued.

Now pick any $\mathcal{A}_\beta \in \mathcal{C} \cap V_{\kappa'}$ and let

$$h_\beta : \prod(\mathcal{C} \cap V_{\kappa'}) \to \mathcal{A}_\beta$$

be the projection map.
Define $k_\beta : V_{\kappa+1} \to V_{\beta+1}$ by:

$$k_\beta(X) = h_\beta(j(\{X \cap V_\alpha\}_{\alpha \in I})).$$

For each $a \in [\beta]^{<\omega}$, define E_a^β by

$$X \in E_a^\beta \quad \text{iff} \quad X \subseteq [\kappa]^{|a|} \text{ and } a \in k_\beta(X)\,.$$

Then E_a^β is an ω_1-complete proper ultrafilter over $[\kappa]^{|a|}$. Hence the ultrapower $\mathrm{Ult}(V, E_a^\beta)$ is well-founded. Furthermore, since j is faithful, if $\beta \in I$, then E_a^β is the principal ultrafilter generated by $\{a\}$. □

Continued.

Let $\mathcal{E}_\beta := \{E^\beta_a : a \in [\beta]^{<\omega}\}$. One can show that $\mathcal{E}_\beta$ is normal and coherent.
Let $M_{\mathcal{E}_\beta}$ be the direct limit of

$$\langle\langle M^\beta_a : a \in [\beta]^{<\omega}\rangle, \langle i^\beta_{ab} : a \subseteq b\rangle\rangle$$

where the i^β_{ab} are the standard projection maps, and let $j_{\mathcal{E}_\beta} : V \to M_{\mathcal{E}_\beta}$ be the corresponding limit elementary embedding.
One can show that $M_{\mathcal{E}_\beta}$ is well-founded. So, let $\pi_\beta : M_{\mathcal{E}_\beta} \to N_\beta$ be the transitive collapse, and let $j_{N_\beta} = \pi \circ j_{\mathcal{E}_\beta} : V \to N_\beta$.
Moreover, $V_\beta \subseteq N_\beta$ and $j_{N_\beta}(\kappa) \geqslant \beta$.
Note that if $\beta > \kappa$, then this implies that $\text{crit}(j_{N_\beta}) \leqslant \kappa$.

□

Continued.

If $\beta \leqslant \beta'$ are in $I' := \{\alpha : \mathcal{A}_\alpha \in V_{\kappa'}\}$, then $E_a^\beta = E_a^{\beta'}$, for every $a \in [\beta]^{<\omega}$ (this uses that j is $\subseteq$-chain preserving!). Hence, for every $\beta < \beta'$ in I', the map

$$k_{\beta,\beta'} : M_{\mathcal{E}_\beta} \to M_{\mathcal{E}_{\beta'}}$$

given by

$$k_{\beta,\beta'}([a, [f]_{E_a}]_{\mathcal{E}_\beta}) = [a, [f]_{E_a}]_{\mathcal{E}_{\beta'}}$$

is well-defined, elementary, and commutes with the embeddings $j_{\mathcal{E}_\beta} : V \to M_{\mathcal{E}_\beta}$ and $j_{\mathcal{E}_{\beta'}} : V \to M_{\mathcal{E}_{\beta'}}$.

Let M be the direct limit of

$$\langle \langle M_{\mathcal{E}_\beta} : \beta \in I' \rangle, \langle k_{\beta,\beta'} : \beta < \beta' \text{ in } I' \rangle \rangle$$

and let $j_M : V \to M$ be the corresponding limit elementary embedding. Let $\pi^M : M \to N$ be the transitive collapse, and let $j_N = \pi^M \circ j_M : V \to N$.

□

Continued.

Let $\xi = \sup(I')$. Note that $\xi \in C^{(n)}$ and $\xi > \kappa$. Then one shows $j_N(\kappa) = \xi$, hence $\text{crit}(j_N) \leqslant \kappa$. But since for $\beta \in I$ the map j_{N_β} is the identity, $\text{crit}(j_N) = \kappa$.

Also, since $V_\xi = \bigcup_{\beta \in I} V_\beta$, and $V_\beta \subseteq N_\beta$ for all $\beta \in I$, it follows that $V_\xi \subseteq N$.

This shows that κ is ξ-superstrong, hence also λ-$C^{(n)}$-superstrong, as wanted. □

Theorem

For every $n \geqslant 1$*, the following are equivalent for any cardinal* κ*:*

1. κ *witnesses* Π_n-SPSR
2. κ *is a* $C^{(n)}$*-globally superstrong cardinal.*

The Globally Superstrong Hierarchy

Table 3[6]

Complexity	SPSR
Σ_1	ZFC
Π_1, Σ_2	Globally Superstrong
Π_2, Σ_3	$C^{(2)}$-Globally Superstrong
Π_3, Σ_4	$C^{(3)}$-Globally Superstrong
$\vdots$	$\vdots$
Π_n, Σ_{n+1}	$C^{(n)}$-Globally Superstrong
$\vdots$	$\vdots$
Π_n, all n	$C^{(n)}$-Globally Superstrong, all n

[6] J. Bagaria. Large Cardinals as Principles of Structural Reflection. *Preprint*.

Beyond Vopěnka's Principle

Joint ongoing work with Philipp Lücke[7]

[7] *Huge Reflection*. Preprint available soon.

Exact $\mathcal{SR}$

Definition

Let $\mathcal{C}$ be a class of structures of the same type.

1. Given infinite cardinals $\lambda > \kappa$, let $\mathrm{ESR}_{\mathcal{C}}(\kappa, \lambda)$ be the assertion that for every $A \in \mathcal{C}$ of rank λ there exists some $B \in \mathcal{C}$ of rank κ and an elementary embedding form B into A.
2. Let $\mathrm{UESR}_{\mathcal{C}}(\kappa)$[8] assert that $\mathrm{ESR}_{\mathcal{C}}(\kappa, \lambda)$ holds for a proper class of cardinals $\lambda > \kappa$.
3. Given a definability class Γ and a class P, let $\Gamma(P)$-$\mathrm{ESR}(\kappa, \lambda)$ be the statement asserting that $\mathrm{ESR}_{\mathcal{C}}(\kappa, \lambda)$ holds for every class $\mathcal{C}$ of structures of the same type that is Γ-definable with parameters in P.
   We shall write $\Gamma(P)^{ic}$-$\mathrm{ESR}(\kappa, \lambda)$ and $\Gamma(P)^{ic}$-$\mathrm{UESR}(\kappa)$ for the statements $\Gamma(P)$-$\mathrm{ESR}(\kappa, \lambda)$ and $\Gamma(P)$-$\mathrm{UESR}(\kappa)$, respectively, restricted to classes $\mathcal{C}$ that are Γ-definable with parameters in P *and* are closed under isomorphic copies.

[8] The "U" is for "Unbounded".

Exact $\mathcal{SR}$

Fact

Let $\kappa < \lambda$ *be inaccessible cardinals and let* $\mathcal{C}$ *be a class of structures of the same type that is closed under isomorphic copies. Then* $\mathrm{ESR}_{\mathcal{C}}(\kappa, \lambda)$ *holds if and only if for every structure* $B \in \mathcal{C}$ *of cardinality* λ*, there exists an elementary embedding of a structure* $A \in \mathcal{C}$ *of cardinality* κ *into* B*.* □

Exact $\mathcal{SR}$

Proposition

If $\mathcal{C}$ is a class of structures of the same type that is closed under isomorphic copies and is definable by a Σ_1-formula with parameter z, then $\mathrm{ESR}_{\mathcal{C}}(\kappa, \lambda)$ holds for all uncountable cardinals $\kappa < \lambda$ with $\operatorname{cof}\kappa \leqslant \operatorname{cof}\lambda$ and such that $z \in V_\kappa$. In particular, if κ is an uncountable cardinal, then $\Sigma_1(V_\kappa)^{ic}\text{-UESR}_{\mathcal{C}}(\kappa)$ holds.

Lemma

If there are uncountable cardinals $\kappa < \lambda$ with the property that $\mathrm{ESR}_{\mathcal{C}}(\kappa, \lambda)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_0-formula without parameters, then $a^\#$ exists for every real a.

Lemma

If δ is a Ramsey cardinal, then the set of inaccessible cardinals $\kappa < \delta$ with the property that $\Sigma_1(V_\kappa)\text{-ESR}(\kappa)$ holds in V_δ is unbounded in δ.

Exact cardinals

Definition

1. Given cardinals $\kappa < \lambda$ and $n < \omega$, the cardinal κ is *n-exact for λ* if for every $A \in V_{\lambda+1}$, there exist $\kappa' \in C^{(n)}$ greater than κ, $\lambda' \in C^{(n+1)}$ greater than λ, $X \preceq V_{\kappa'}$ with $V_\kappa \cup \{\kappa\} \subseteq X$, and an elementary embedding $j : X \to V_{\lambda'}$ with $j(\kappa) = \lambda$ and $A \in \mathrm{ran}(j)$. If we further require that $j(\mathrm{crit} j) = \kappa$ holds, then we say that κ is *parametrically n-exact for λ*.
2. A cardinal κ is *n-exact* (respectively, *parametrically n-exact*) if it is n-exact (respectively, parametrically n-exact) for a proper class of cardinals λ.

Theorem

The following are equivalent:

1. κ *is the least* n*-exact cardinal for* λ.
2. κ *is the least parametrically* n*-exact cardinal for* λ.
3. κ *is the least cardinal for which* $\Sigma_{n+1}\text{-ESR}(\kappa, \lambda)$ *holds.*

Huge Reflection

Recall that a cardinal κ is *huge* if there exists an elementary embedding $j : V \to M$, with M transitive, such that $\kappa = \text{crit}(j)$ and ${}^{j(\kappa)}M \subseteq M$.

Also, recall that a cardinal κ is *almost huge* if there exists an elementary embedding $j : V \to M$, with M transitive, such that $\kappa = \text{crit}(j)$ and ${}^{\gamma}M \subseteq M$ for every $\gamma < j(\kappa)$.

Huge Reflection

Theorem

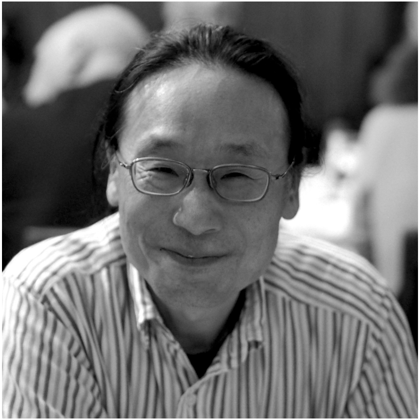
If κ is a huge cardinal, witnessed by j, then κ is parametrically n-exact for $j(\kappa)$, for every n. Hence, $\mathrm{ESR}_{\mathcal{C}}(\kappa, j(\kappa))$ holds for all classes $\mathcal{C}$ of structures of the same type that are definable with parameters in V_κ.

Corollary

If κ is superhuge, then κ is n-parametrically exact, for every n. Hence, $\mathrm{UESR}_{\mathcal{C}}(\kappa)$ holds for all classes of structures $\mathcal{C}$ of the same type that are definable with parameters in V_κ.

Theorem

If κ is parametrically 0-exact for λ, then κ is almost-huge with target λ. Moreover, the set of almost-huge cardinals with target κ is stationary in κ, and the set of almost-huge cardinals with target λ is stationary in λ.



Congratulations!