On relative definability

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Focus: To quantify the idea that a certain combinatorial object of size $\aleph_1$ necessarily encodes more information than a certain other type of object of size $\aleph_1$.

Example: If $\vec{C} = (C_\delta : \delta \in \mathrm{Lim}(\omega_1))$ is a ladder system, then from $\vec{C}$ we can always define a Hausdorff gap, a simplified $(\omega, 1)$-morass, an Aronszajn tree, a Countryman line, a partition of ω_1 into $\aleph_1$-many stationary sets, etc. All these claims can be verified by looking at specific constructions of the relevant objects.

On the other hand, it need not be the case that if $\vec{S} = (S_\alpha : \alpha < \omega_1)$ is a partition of ω_1 into stationary stationary, then there is a ladder system $\vec{C}$ definable from $\vec{S}$.

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Typical proof of such separation results:

(1) Start with a model V satisfying a suitable existence-pattern of the relevant type of objects at some cardinal $\kappa > \omega_1$.
(2) Pick a sample of objects that exist there.
(3) Argue, in some homogeneous extension in which κ becomes ω_1, that the objects from the sample have been preserved and, moreover, no objects of unwanted kind are definable from them as otherwise they would be in V by homogeneity of the forcing, which is not the case for the V we started with.

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Many of these separation results regarding definability (but not all of them) can be easily turned into independence results relative to ZF by going to a natural symmetric submodel of the generic extension in (3).

Definition

Given two properties $P(x)$, $Q(x)$, $Q(x)$ *has definability strength at least that of* $P(x)$ *over* $\langle H(\omega_2), \in\rangle$ if there is a formula $\varphi(x, y)$ such that

$$P(\{b \in H(\omega_2) \,:\, H(\omega_2) \models \varphi(A, b)\})$$

for every $A \in H(\omega_2)$ such that $Q(A)$.

This study becomes of course trivial if there is a well-order $\leq$ of $H(\omega_2)$ definable over $H(\omega_2)$ without parameters: Given any property $P(x)$, simply look for its $\leq$-first witness.

One can of course modify this definition.

- One can restrict φ to having some bounded complexity (e.g. Σ_1 or Σ_0).
- One can look at *definability strength over* V, or *definability strength over* $H(\omega_2)$ *with real parameters*, etc., in the obvious way.

Example:

Definition

Given two properties $P(x)$, $Q(x)$, $Q(x)$ *has definability strength at least that of* $P(x)$ *over* $\langle H(\omega_2), \in \rangle$ *with real parameters* if there is a formula $\varphi(x, y, z)$ and a real r such that

$$P(\{b \in H(\omega_2) \ : \ H(\omega_2) \models \varphi(A, b, r)\})$$

for every $A \in H(\omega_2)$ such that $Q(A)$.

Proposition

*(Woodin) The $\mathbb{P}_{max}$ axiom $(*)$ implies that if $P(x)$ is **any** property definable over $(H(\omega_2), \in)$ such that $P(x)$ for some $x \in H(\omega_2)$, then*

"x is a subset of ω_1 not constructible from a real"

has definability strength at least that of $P(x)$ over $\langle H(\omega_2), \in\rangle$ with real parameters.

Proof.

If $(*)$ holds and $A \in \mathcal{P}(\omega_1)$ is such that $A \notin L[r]$ for any real r, then $L_{\omega_2}(\mathbb{R})[A] = H(\omega_2)$.

In fact, for every $B \in H(\omega_2)$ there is a $\mathbb{P}_{max}$-condition (M, I, a) and some $b \in M$ such that there is a unique correct generic iteration of (M, I, a) sending a to A. Moreover, if $j : (M, I) \longrightarrow (M^*, I^*)$ is the corresponding map, then $j(b) = B$. □

Similarly:

Proposition

*Suppose BMM holds and NS_{ω_1} is saturated. If $P(x)$ is **any** property definable over $(H(\omega_2), \in)$ such that $P(x)$ for some $x \in H(\omega_2)$, then*

"x is a stationary and co-stationary subset of ω_1"

has definability strength at least that of $P(x)$ over $\langle H(\omega_2), \in \rangle$ with real parameters.

Proof.

Suppose BMM holds and NS_{ω_1} is saturated. If $B \subseteq \omega_1$ and $S \subseteq \omega_1$ is stationary and co-stationary, then there is a real r such that $B \in L_{\omega_2}[r, S]$.

The proof of this uses the fact that under the hypothesis,

- ψ_{AC} holds (Larson, Woodin),
- every club of ω_1 contains a club constructible from a real (Woodin), and
- if $\alpha < \omega_2$, then there is a real r such that $|\alpha|^{L[r]} = \kappa$, where $\kappa = \omega_1^V$ (Woodin).

□

Remark: MM^{++} implies $(*)$ (A.-Schindler) but BMM + "NS_{ω_1} is saturated" does not imply $(*)$. In fact MM does not imply $(*)$ (Larson).

A high version of Ψ_{AC}

ψ_{AC}: For all stationary and co-stationary S, $T \subseteq \omega_1$ there is some $\alpha < \omega_2$ such that for every surjection $\pi : \omega_1 \longrightarrow \alpha$ there is a club $C \subseteq \omega_1$ such that for every $\nu \in C$, $\nu \in S$ iff $\mathrm{ot}(\pi``\nu) \in T$.

ψ_{AC} implies

- every function $f : \omega_1 \longrightarrow \omega_1$ is dominated mod. a club by the canonical function of some $\alpha < \omega_2$;
- if $\vec{S} = (S_\alpha \ : \ \alpha < \omega_1)$ is a partition of ω_1 into stationary sets, then there is a well-order of $H(\omega_2)$ definable over $H(\omega_2)$ from $\vec{S}$.

Also: ψ_{AC} can be forced starting from an inaccessible limit of measurable cardinals.

Consider the following high version of ψ_{AC}.

Definition

$\psi_{AC}^{S^{\omega_2}_{\omega_1}}$: For all stationary $S, T \subseteq S^{\omega_2}_{\omega_1}$ with $S^{\omega_2}_{\omega_1} \setminus S$ and $S^{\omega_2}_{\omega_1} \setminus T$ also stationary there is some $\alpha < \omega_3$ such that for every surjection $\pi : \omega_2 \longrightarrow \alpha$ there is a club $C \subseteq \omega_2$ such that for every $\nu \in C \cap \mathrm{cf}(\omega_1)$, $\nu \in S$ iff $\mathrm{ot}(\pi``\nu) \in T$.

ψ_{AC} implies

- every function $f : S^{\omega_2}_{\omega_1} \longrightarrow \omega_2$ is dominated mod. a club by the canonical function of some $\alpha < \omega_3$;
- if $\vec{S} = (S_\alpha : \alpha < \omega_2)$ is a partition of $S^{\omega_2}_{\omega_1}$ into stationary sets, then there is a well-order of $H(\omega_3)$ definable over $H(\omega_3)$ from $\vec{S}$.

In current work in progress with Veličković (currently being written up), and starting with an inaccessible limit of measurables, we are building a forcing notion with (hopefully) forces $\psi_{AC}^{S_{\omega_1}^{\omega_2}}$. The construction involves side conditions of virtual models of two types.

Form now on I will focus mostly on the original (lightface) notion of definability strength.

Interdefinability

Theorem

(ZF) The following properties are interdefinable over $\langle H(\omega_2), \in \rangle$.

(1) x *is a ladder system*

(2) x *is a simplified* $(\omega, 1)$*-morass*

(3) x *is an special Aronszajn tree with a witness*

(4) x *is a Countryman line with a witness*

(5) x *is an indestructible gap with a witness*

Specific constructions show that $(1) \Rightarrow (2), (3), (4), (5)$.

To see, for example, that $(5) \Rightarrow (1)$, suppose $\kappa = \omega_1$ and $A \subseteq \omega_1$ canonically codes some

$$(L = (\kappa, \leq), (C_n)_{n<\omega}),$$

where $L = (\kappa, \leq)$ is a Countryman line as witnessed by a decomposition $L \times L = \bigcup_{n<\omega} C_n$ into chains C_n. One can then prove that $\omega_1^{L[A]} = \kappa$, and so in $L_{\omega_2}[A]$ we may pick the $<_{L[A]}$-least ladder system on κ. $\square$

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Remark: Ladder systems certainly need not define everything interesting that may exist in $H(\omega_2)$:

Suppose $\vec{C}$ is a ladder system and Weak Club Guessing fails. Add a Cohen real c. In $H(\omega_2)^{V[c]}$ there is a Weak Club Guessing ladder system $\vec{D}$, but no such $\vec{D}$ can be definable from $\vec{C}$ as otherwise $\vec{D} \in V$ by homogeneity of Cohen forcing.

Compare with situation under $(*)$, in which everything not in $L(\mathbb{R})$ can define a witness for every true definable property.

An aside: One can run the same argument with other (more interesting?) club-guessing properties. For example:

Interval Hitting Principle (IHP):

There is a ladder system $(C_\delta : \delta \in \mathrm{Lim}(\omega_1))$ such that for every club $C \subseteq \omega_1$ there is some δ such that

$$[C_\delta(n), C_\delta(n+1)) \cap C \neq \emptyset$$

for a tail of $n < \omega$.

Theorem

There is a c.c.c. homogeneous forcing $\mathcal{P}$ definable over $H(\omega_2)$ without parameters and which forces IHP.

Conditions p in $\mathcal{P}$ have

- a working part c_p, which is a finite function with $\mathrm{dom}(c_p) \subseteq \mathrm{Lim}(\omega_1)$ such that $c_p(\delta) \in [\delta]^{<\omega}$ for each $\delta \in \mathrm{dom}(c_p)$, and
- a side condition $\mathcal{D}_p$, which is a finite function with $\mathrm{dom}(\mathcal{D}_p) \subseteq \mathrm{Lim}(\omega_1)$ such that for each $\delta \in \mathrm{dom}(\mathcal{D}_p)$, $\mathcal{D}_p(\delta)$ is a finite set of cofinal subsets of δ.

On the same vein:

Question: Is there a homogeneous forcing definable over $H(\omega_2)$ without parameters and which forces Club Guessing?

Separation results

Theorem

(1) It is consistent that there is an Aronszajn tree T, an (ω_1, ω_1^*)-gap $(\vec{f}, \vec{g})$ in $({}^\omega\omega, <^*)$, and a partition $\vec{S}$ of ω_1 into $\aleph_1$-many stationary sets such that no ladder system is definable from $(T, (\vec{f}, \vec{g}), \vec{S})$.

(2) If there is an inaccessible cardinal, then the following holds in a symmetric submodel of a forcing extension of V: There is an Aronszajn tree, an (ω_1, ω_1^*)-gap in $({}^\omega\omega, <^*)$, and a partition of ω_1 into $\aleph_1$-many stationary sets, but there is no ladder system on ω_1.

(3) It is consistent that there is an Aronszajn tree T, an ω_1-sequence $\vec{r}$ of distinct reals, and a partition $\vec{S}$ of ω_1 into $\aleph_1$-many stationary sets such that no (ω_1, ω_1^*)-gap in $({}^\omega\omega, <^*)$ is definable from $(T, \vec{r}, \vec{S})$.

(4) If there is an inaccessible cardinal, then the following holds in a symmetric submodel of a forcing extension of V: There is an Aronszajn tree, an ω_1-sequence of distinct reals, and a partition of ω_1 into $\aleph_1$-many stationary sets but there is no (ω_1, ω_1^*)-gap in $({}^\omega\omega, <^*)$.

(5) It is consistent, relative to ZFC, that there is an Aronszajn tree T and a partition $\vec{S}$ of ω_1 into $\aleph_1$-many stationary sets such that no ω_1-sequence of distinct reals is definable from $(T, \vec{S})$.

(6) If there is an inaccessible cardinal, then the following holds in a symmetric submodel of a forcing extension of V: There is an Aronszajn tree and a partition of ω_1 into $\aleph_1$-many stationary sets but there is no ω_1-sequence of distinct reals.

Remark: The use of an inaccessible in the even-numbered parts is necessary.

Proof of (1): Start with a model with an $\aleph_2$-Aronszajn tree T. Let $\vec{S} = (S_\nu)_{\nu<\omega_2}$ be any partition of ω_2 into stationary sets. Let $\mathcal{P}$ be c.c.c. forcing for adding (ω_1, ω_1^*)-gap, let G be $\mathcal{P}$-generic and let $(\vec{f}, \vec{g})$ be the generic gap added by G.

Claim

Every c.c.c. forcing $\mathcal{Q}$ preserves the Aronszajnness of T. In particular, T is Aronszajn in $V[G]$.

Proof.

Otherwise there is a $\mathcal{Q}$-name $\dot{b}$ for a cofinal branch through T and a subtree $T' \subseteq T$ of height ω_2 with countable levels such that $\Vdash_{\mathcal{Q}} \dot{b} \restriction \alpha \in T'_\alpha$ for every $\alpha < \omega_2$. But for every regular κ and $\lambda < \kappa$, every tree of height κ with levels of size less than λ has a κ-branch, and so T', and therefore also T, has an ω_2-branch, which is a contradiction. $\square$

In $V[G]^{\mathrm{Coll}(\omega,\omega_1)}$, T remains Aronszajn, every S_ν remains a stationary subset of $\omega_2^{V[G]} = \omega_2^V$, $\omega_1 = \omega_2^V$, and $(\vec{f}, \vec{g})$ is still a gap: Suppose G' is $\mathrm{Coll}(\omega, \omega_1)$-generic over $V[G]$ and r is a real in $V[G][G']$. Then $r \in V[G \restriction \alpha][G']$ for some $\alpha < \omega_2$. But then r cannot split $(\vec{f}, \vec{g})$.

Finally, in $V[G]^{\mathrm{Coll}(\omega,\omega_1)}$ there cannot be any ladder system on $\omega_1 = \omega_2^{V[G]}$ definable from $(T, (\vec{f}, \vec{g}), \vec{S})$. Otherwise, by homogeneity of the collapse this ladder system would be in $V[G]$, which is impossible.

Proof of (2): Let κ be an inaccessible cardinal such that there is a κ-Aronszajn tree T, let $(S_\nu)_{\nu<\kappa}$ be a partition of κ into stationary sets, and let G be generic for natural c.c.c. forcing $\mathcal{P}_\kappa^\kappa$ for adding (κ, κ^*)-gap. Our model W will be the symmetric submodel of the extension of $V[G]$ by $\mathrm{Coll}(\omega, <\kappa)$ generated by the names fixed by an automorphism of $\mathrm{Coll}(\omega, <\kappa)$ fixing $\mathrm{Coll}(\omega, <\alpha)$ for some $\alpha < \kappa$.

In W, every α is collapsed to ω and so $\omega_1 = \kappa$, each S_ν is clearly stationary, T remains Aronszajn, and $(\vec{f}, \vec{g})$ is a (κ, κ^*)-gap.

In W there is no ladder system on ω_1 as such an object would be in $V[G][H]$ for some $\mathrm{Coll}(\omega, <\alpha)$-generic H for some $\alpha < \kappa$, which is impossible. $\square$

Theorem

Suppose there is a weakly compact cardinal. Then there is a partial order $\mathcal{P}$ such that

(1) *$\mathcal{P}$ forces that there is a partition $\vec{S}$ of ω_1 into $\aleph_1$-many stationary sets and an (ω_1, ω_1^*)-gap $(\vec{f}, \vec{g})$ in $({}^\omega\omega, <^*)$ such that no Aronszajn tree is definable from $(\vec{S}, (\vec{f}, \vec{g}))$, and*

(2) *there is a symmetric submodel of a forcing extension by $\mathcal{P}$ satisfying that there is a partition of ω_1 into $\aleph_1$-many stationary sets, an (ω_1, ω_1^*)-gap in $({}^\omega\omega, <^*)$, but no Aronszajn tree.*

Proof:

Let κ be weakly compact, let $\vec{S}$ be a partition of κ into κ-many stationary sets, let G be $\mathcal{P}_\kappa^\kappa$–generic and $(\vec{f}, \vec{g})$ the corresponding generic gap, let M be a transitive model of enough of ZFC containing everything relevant and such that $V_\kappa \subseteq M$ and $|M| = \kappa$, and let $j : M \longrightarrow N$ be an elementary embedding, N transitive, with $\mathrm{crit}(j) = \kappa$.

Then j can be extended to elementary embedding

$$j : M[G] \longrightarrow N[G][H],$$

where H is $(\mathcal{P}^{j(\kappa)}_{j(\kappa)})^N/G$-generic over $N[G]$ (since $(\mathcal{P}^{\kappa}_{\alpha})_{\alpha\leq\kappa}$ is the initial segment of $(\mathcal{P}^{j(\kappa)}_{\alpha})^N_{\alpha<j(\kappa)}$ of length κ).

Now, every κ-tree T in $M[G]$ acquires a κ-branch in $N[G][H]$ since $j(T) \upharpoonright \kappa = T$. But $(\mathcal{P}^{j(\kappa)}_{j(\kappa)})^N/G$ has the c.c.c. in $N[G]$, and therefore T already had a κ-branch in $N[G]$ by previous Claim.

Hence there are no κ-Aronszajn trees after forcing with $\mathcal{P}^{\kappa}_{\kappa}$. Now it is easy to see, as in the proof of previous Theorem, that $\mathcal{P} = \mathcal{P}^{\kappa}_{\kappa} * \mathrm{Coll}(\omega, <\kappa)$ is as desired. $\square$

Remark: The weakly compact cardinal is necessary for part (2) by a classical result of Silver. I don't know if the weakly compact cardinal is necessary for part (1), though.

Partitioning ω_1 into stationary sets

Fact

(1) *(ZF + AC_ω) If $\vec{C}$ is a ladder system, then there is a partition of ω_1 into $\aleph_1$-many stationary sets definable over $H(\omega_2)$ from $\vec{C}$.*

(2) *(ZF+ AC_ω) If $\vec{r} = (r_\alpha)_{\alpha<\omega_1}$ is a one-to-one ω_1-sequence of reals, then there is a partition of ω_1 into $\aleph_0$-many stationary sets definable over $H(\omega_2)$ from $\vec{r}$.*

Question: Does ZF prove that if $\vec{C}$ is a ladder system, then there is a stationary and co-stationary subset of ω_1 definable from $\vec{C}$?

Theorem

Let $\lambda \leq \omega$, $0 < \lambda$. The following theories are equiconsistent.

(1) ZFC + *There is a measurable cardinal.*

(2) ZFC + *There is a partition $(S_i)_{i<\lambda}$ of ω_1 into stationary sets such that no partition of ω_1 into more than λ-many stationary sets is definable from $(S_i)_{i<\lambda}$.*

Proof: (1) $\Rightarrow$ (2): Let κ be measurable. By Kunen-Paris we may assume there are distinct normal measures $\mathcal{U}_i$ on κ for $i < \lambda$. Pick stationary $S_i \subseteq \kappa \cap \mathsf{Inacc}$, for $i < \lambda$, such that for all $i^* < \lambda$, i^* is the unique $i < \lambda$ such that $S_{i^*} \in \mathcal{U}_i$.

In $V^{\mathrm{Coll}(\omega, <\kappa)}$, let $\dot{\mathcal{P}}$ be a homogeneous forcing preserving the stationarity of all S_i and adding a club C of $\kappa = \omega_1$, $C \subseteq \bigcup_{i<\lambda} S_i$, diagonalizing all $\mathcal{U}_i$ (i.e., $\dot{\mathcal{P}}$ also adds enumerations $(X^i_\alpha)_{\alpha<\kappa}$ of $\mathcal{U}_i$ for each $i < \lambda$ such that for all $\alpha \in C \cap S_i$, $\alpha \in \bigcap_{\beta<\alpha} X^i_\beta$).

Now let H be $\mathrm{Coll}(\omega, <\kappa) * \dot{\mathcal{P}}$-generic over V, let C be the generic club of κ added by $\dot{\mathcal{P}}$ over $V^{\mathrm{Coll}(\omega, <\kappa)}$, and suppose, towards a contradiction, that there is a cardinal $\lambda' > \lambda$ and a partition $(A_i)_{i<\lambda'}$ of $\omega_1^{V[H]} = \kappa$ into stationary sets definable from $(S_i)_{i<\lambda}$.

By homogeneity of $\mathrm{Coll}(\omega, <\kappa) * \dot{\mathcal{P}}$, $(A_i)_{i<\lambda'} \in V$.

There must then be some $i^* < \lambda$ and two distinct $i_0, i_1 < \lambda'$ such that both $A_{i_0} \cap S_{i^*}$ and $A_{i_1} \cap S_{i^*}$ are stationary in $V[H]$. If, say, $A_{i_0} \cap S_{i^*} \notin \mathcal{U}_{i^*}$, then $A_{i_0} \cap S_{i^*}$ is non-stationary since C diagonalizes $\mathcal{U}_{i^*}$. Contradiction.

(2) $\Rightarrow$ (1): Let A be a set of ordinals definable from, and coding $(S_i)_{i<\lambda}$. Using a standard tree-splitting argument we show that $\kappa = \omega_1$ is measurable in the ZFC-model $\mathrm{HOD}(A)$. $\square$

A very weak guessing principle interdefinable with a stationary and co-stationary subset of ω_1

Definition

(ZF) $\vec{X} = (X_\delta : \delta \in \mathrm{Lim}(\omega_1))$ is a *very weak guessing sequence* if

(1) $X_\delta \subseteq \delta$ for each δ.

(2) For each club $C \subseteq \omega_1$,

  (a) there is some $\delta \in C$ such that $C \cap X_\delta \neq \emptyset$, and
  (b) there is some $\delta \in C$ such that $(C \cap \delta) \setminus X_\delta \neq \emptyset$.

Let $\mathcal{C}_{\omega_1} = \{X \subseteq \omega_1 : C \subseteq X \text{ for some club } C \text{ of } \omega_1\}$.

Theorem

(ZF+ $\mathcal{C}_{\omega_1}$ is a normal filter on ω_1)

(1) *Suppose $S \subseteq \omega_1$ is stationary and co-stationary. Then there is a very weak guessing sequence $\vec{X}$ definable over $H(\omega_2)$ from S.*

(2) *Suppose $\vec{X}$ is a very weak guessing sequence. Then there is a stationary and co-stationary $S \subseteq \omega_1$ definable over $H(\omega_2)$ from $\vec{X}$.*

Proof:

Suppose $S \subseteq \omega_1$ is stationary and co-stationary. Then

$$(S \cap \delta : \delta \in \mathrm{Lim}(\omega_1))$$

is a very weak guessing sequence.

Now suppose $\vec{X} = (X_\delta : \delta \in \text{Lim}(\omega_1))$ is a very weak guessing sequence. We have two possible cases.

Case 1: For all $\alpha < \omega_1$, either

- $W^0_\alpha = \{\delta < \omega_1 : \alpha \notin X_\delta\}$ is in $\mathcal{C}_{\omega_1}$, or
- $W^1_\alpha = \{\delta < \omega_1 : \alpha \in X_\delta\}$ is in $\mathcal{C}_{\omega_1}$.

For each $\alpha < \omega_1$ let W_α be W^ϵ_α for the unique $\epsilon \in \{0, 1\}$ such that $W^\epsilon_\alpha \in \mathcal{C}_{\omega_1}$, and let

$$W^* = \Delta_{\alpha<\omega_1} W_\alpha \in \mathcal{C}_{\omega_1}$$

Then $X_{\delta_0} = X_{\delta_1} \cap \delta_0$ for all $\delta_0 < \delta_1$ in W^*. It then follows that

$$S = \bigcup_{\delta \in W^*} X_\delta,$$

which of course is definable from $\vec{X}$, is a stationary and co-stationary subset of ω_1.

Indeed, suppose $C \subseteq \omega_1$ is a club, and let us fix a club $D \subseteq W^*$. There is then some $\delta \in C \cap D$ and some $\alpha \in C \cap D \cap X_\delta$. But then $\alpha \in S$ since $\delta \in W^*$ and $\alpha \in W^* \cap X_\delta$. There is also some $\delta \in C \cap D$ and some $\alpha \in C \cap D$ such that $\alpha \notin X_\delta$, which implies that $\alpha \notin S$ by a symmetrical argument, using the fact that $X_{\delta_0} = X_{\delta_1} \cap \delta_0$ for all $\delta_0 < \delta_1$ in W^*.

Case 2: There is some $\alpha < \omega_1$ with the property that both W^0_α and W^1_α are stationary subsets of ω_1. But now we can let S be W^0_α, where α is first such that W^0_α is stationary and co-stationary. $\square$

Recall:

Theorem

Let $\lambda \leq \omega$, $0 < \lambda$. The following theories are equiconsistent.

(1) ZFC + *There is a measurable cardinal.*

(2) ZFC + *There is a partition $(S_i)_{i<\lambda}$ of ω_1 into stationary sets such that no partition of ω_1 into more than λ-many stationary sets is definable from $(S_i)_{i<\lambda}$.*

I don't know how to convert the proof of previous Theorem into a corresponding consistency result over ZF. However, one can prove such results starting with a model of $\mathsf{ZF} + \mathsf{AD}$. For example, a classical well-known result of Solovay is that, under AD, the club filter on $\delta^1_1 = \omega_1$ is an ultrafilter and therefore ω_1 cannot be partitioned into 2 stationary sets. The following theorem generalises this.

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Let $\lambda \leq \omega$, $0 < \lambda$. The following theories are equiconsistent.

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Recall that, for a nonzero $n \in \omega$, δ^1_n denotes the supremum of the lengths of all $\mathbf{\Delta}^1_n$–pre-wellorderings of the reals.

Theorem

($\mathsf{ZF} + \mathsf{AD}$) For every $n < \omega$, δ^1_{2n+1} is a successor cardinal and regular and, letting κ be such that $\kappa^+ = \delta^1_{2n+1}$, $\mathrm{Coll}(\omega, \kappa)$ forces that there is a partition of $(\delta^1_{2n+1})^V = \omega_1$ into $2^{n+1} - 1$ stationary sets but no partition of ω_1 into more than $2^{n+1} - 1$ stationary sets.

Proof:

Let $\lambda = \boldsymbol{\delta}^1_{2n+1}$. By a result of Kechris, λ is a successor cardinal.

For every infinite regular cardinal $\mu < \lambda$, the $\mathcal{C}^\lambda_\mu$ be the filter generated by the μ-closed and unbounded subsets of λ. By a result of Jackson there are exactly $2^{n+1} - 1$ many infinite regular cardinals κ below λ and λ has the strong partition property (i.e., $\lambda \longrightarrow (\lambda)^\lambda_\alpha$ for all $\alpha < \lambda$). This guarantees that λ is measurable and that, moreover,

$$\{\mathcal{C}^\lambda_\mu \,:\, \mu < \lambda, \mu \text{ an infinite regular cardinal}\}$$

is the set of normal measures on λ.

Let now κ be such that $\kappa^+ = \lambda$ and note that $\mathrm{Coll}(\omega, \kappa)$ can be well-ordered in length κ. It follows that $\mathrm{Coll}(\omega, \kappa)$ forces $\omega_1 = \lambda$. It follows also that every club of λ in any extension by $\mathrm{Coll}(\omega, \kappa)$ contains a club in V, and hence $\mathrm{Coll}(\omega, \kappa)$ preserves the stationarity of $\lambda \cap \mathrm{cf}^V(\mu)$ for every infinite V-regular $\mu < \lambda$.

Finally, let $\dot{S}$ be a name for a subset of $\lambda \cap \mathrm{cf}^V(\mu)$, for some such μ, let $\bar{p} \in \mathrm{Coll}(\omega, \kappa)$, and note that $\bar{p}$ forces

$$\dot{S} \subseteq S^* := \bigcup_{p \in \mathrm{Coll}(\omega, \kappa) \restriction \bar{p}} \{\alpha \in \mathrm{cf}^V(\mu) : p \Vdash_{\mathrm{Coll}(\omega, \kappa)} \alpha \in \dot{S}\}$$

Again by the fact that $\mathrm{Coll}(\omega, \kappa)$ can be well-ordered in length κ, it follows that there is a μ-club $D \in V$ such that either

$$D \cap S^* = \emptyset$$

or

$$D \subseteq \{\alpha \in \mathrm{cf}(\mu) : p \Vdash_{\mathrm{Coll}(\omega, \kappa)} \alpha \in \dot{S}\}$$

for some $p \leq \bar{p}$. From this we immediately get that $\mathrm{Coll}(\omega, \kappa)$ implies that there is no partition of κ into more than $2^{n+1} - 1$ many stationary sets. $\square$

Question: Let $m \in \omega$ be not of the form $2^{n+1} - 1$ for any $n < \omega$. Is it consistent with ZF to have a partition of ω_1 into m stationary sets but no partition of ω_1 into more than m stationary sets?

Theorem

(Todorčević) BMM implies that if $\vec{r} = (r_\alpha)_{\alpha<\omega_1}$ is a one-to-one enumeration of reals, then there is a well-order of $H(\omega_2)$ definable over $(H(\omega_2), \in)$ from $\vec{r}$.

Theorem

(Moore) BPFA implies that if $\vec{C}$ is a ladder system, then there is a well-order of $H(\omega_2)$ definable over $(H(\omega_2), \in)$ from $\vec{C}$.

Question: Does BPFA imply that if $\vec{r} = (r_\alpha)_{\alpha<\omega_1}$ is a one-to-one enumeration of reals, then there is a well-order of $\mathbb{R}$ definable over $(H(\omega_2), \in)$ from $\vec{r}$?

Question: Suppose MM holds. Does it follow that if $P(x)$ is any property definable over $(H(\omega_2), \in)$, then

"x is a subset of ω_1 not constructible from a real"

has definability strength at least that of $P(x)$ over $\langle H(\omega_2), \in\rangle$ with real parameters?

Does it follow that if $A \subseteq \omega_1$ is not constructible from a real and $B \subseteq \omega_1$, then there is a real r such that $B \in L_{\omega_2}[r, A]$?

Question: Is it possible to have the following simultaneously?

(1) Large cardinals ($X^\sharp$ exists for every $X \subseteq \omega_1$).

(2) There are exactly two definability degrees over $H(\omega_3)$ for definitions with parameters in $\mathcal{P}(\omega_1)$ (in the same way that $(*)$ implies that there are exactly two definability degrees over $H(\omega_2)$ for definitions with parameters in $\mathbb{R}$, viz. those properties exemplified in $L(\mathbb{R})$ and those not exemplified in $L(\mathbb{R})$).

Congratulations, Sakae, no more admin!