ENDOMORPHISMS OF A_n AND ASSOCIATED PROJECTIVE MODULES OF RANK 1.

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Let k be a field of characteristic $p \neq 0$. Let n be a positive integer. A Weyl algebra $A_n(k)$ over a commutative ring k is an algebra over k generated by $2n$ elements $\{\gamma_1, \gamma_2, \dots, \gamma_{2n}\}$ with the "canonical commutation relations"

$$\text{(CCR)} \qquad [\gamma_i, \gamma_j](= \gamma_i\gamma_j - \gamma_j\gamma_i) = h_{ij} \qquad (1 \leq i, j \leq 2n),$$

where h is a non-degenerate anti-Hermitian $2n \times 2n$ matrix of the following form:

$$(h_{ij}) = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}.$$

Assume we have an endomorphism φ of A_n. We have already shown [2][3] that we may decompose φ into an endomorphism f_φ of affince scheme $\operatorname{Spec}(Z_n) = \mathbb{A}^{2n}$ and a conjugation by a matrix valued function g_φ:

$$\varphi(x) = g_\varphi(f_\varphi^* x) g_\varphi^{-1}$$

To make arguments simpler, we assume further that the total degree $\deg(\varphi)$ of φ is "low", that means, it is less than $p/2$. The function g_φ may be considered as a parallel section

$$g_\varphi \in \operatorname{Hom}(f_\varphi^* V, V)^{\nabla^{\text{gauge}}}$$

of a connection ∇. The "integrability" of ∇^{gauge} is equivalent to the condition that φ preserves the symplectic form

$$\omega = \sum_{ij} dT_i dT_j.$$

Now we would like to consider a converse problem: given a symplectic map $f : \mathbb{A}^{2n} \to \mathbb{A}^{2n}$ with a "low" degree, can we construct φ such that $f_\varphi = f$? We may consider a module

$$W_f = \operatorname{Hom}(f^* V, V)^{\nabla}.$$

Proposition 0.1. *W_f is a projective A_n-module of rank one. f is liftable to an endomorphism of A_n if and only if W_f is free.*

It is known that the projective $A_n(\mathbb{C})$-modules of rank one are parametrized by Calogero-Moser spaces [1].

One may consider

QUESTION 0.2. When is W_f free?

In particular, one may ask:

QUESTION 0.3. (With a suitable definition of "homotopy"), if W is homotopic to a free A-module, then is it always free?

Our definition of homotopy of A_n-modules W_t is that it can be considered as an $A_{n+1} = A_n\langle t, \partial_t\rangle$-module.

It turned out that (with our definition of homotopy) the answer to Question 0.3 is false.

EXAMPLE 0.4. Let $A_2 = k\langle \xi, \eta, t, \partial_t\rangle$ be an Weyl algebra over a field k. The commutation relations are given by

$$[\eta, \xi] = 1, [\partial_t, t] = 1,$$

and several other relations which simply says that "irrelevant" variables commute. Then a left A_2-module

$$J = A_2 \cdot (\eta + t(\xi\eta - 1)) + A_2 \cdot \eta^2$$

is projective.

To obtain such examples it may be useful to consider "invariant norms" and $\mathcal{O}_X$-reflexivity of W.

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