Definable Morse functions in a real closed field

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Ordered fields

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  (2) For any $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \boldsymbol{R}$, if $\boldsymbol{x} < \boldsymbol{y}$ and $\boldsymbol{z} > \boldsymbol{0}$, then $\boldsymbol{x}\boldsymbol{z} < \boldsymbol{y}\boldsymbol{z}$.

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  A real field $(R, +, \cdot, <)$ is a *real closed field* if it satisfies one of the following two equivalent conditions.
  (1) [Intermediate value property] For every $f(x) \in R[x]$, if $a < b$ and $f(a) \neq f(b)$, then $f([a, b]_R)$ contains $[f(a), f(b)]_R$ if $f(a) < f(b)$ or $[f(b), f(a)]_R$ if $f(b) < f(a)$, where $[a, b]_R = \{x \in R | a \leq x \leq b\}$.
  (2) The ring $R[i] = R[x]/(x^2 + 1)$ is an algebraically closed field.

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  (3) $\mathbf{R}_{an}^S := (\mathbb{R}, +, \cdot, <, (f), (x^r)_{r \in S})$, where S is a subset of $\mathbb{R}$, f ranges over all restricted analytic functions as in (2), and the function $x^r : \mathbb{R} \to \mathbb{R}$ is given by

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  (5) $\mathbf{R}_{an,exp} := (\mathbb{R}, +, \cdot, <, (f), exp)$, where (f) and exp denote as above.

O-minimal structures

- An ordered structure $(R, <)$ with a dense linear order $<$ without endpoints is *o-minimal (order minimal)* if every definable set of R is a finite union of open intervals and points, where open interval means $(a, b), -\infty \le a < b \le \infty$.

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  If $(\boldsymbol{R}, +, \cdot, <)$ is a real closed field, then it is o-minimal and the collection of definable sets coincides that of semialgebraic sets.
  The topology of $\boldsymbol{R}$ is the interval topology and the topology of $\boldsymbol{R}^n$ is the product topology.
  In this presentation, everything is considered in an o-minimal expansion $\mathcal{N} = (\boldsymbol{R}, +, \cdot, <, \dots,)$ of a real closed field $(\boldsymbol{R}, +, \cdot, <)$ unless otherwise stated.

Definable C^r diffeomorphisms

- Let $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ be definable open sets and $2 \leq r < \infty$. A C^r map $f : U \to V$ is a *definable C^r map* if the graph of f is definable.

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  A C^r diffeomorphism between definable open sets is a *definable C^r diffeomorphism* if its graph is definable.

Definable C^r manifolds and affine definable C^r manifolds

Definition

A Hausdorff space X is an *n-dimensional definable C^r manifold* if there exist a **finite** open cover $\{U_\lambda\}_{\lambda \in \Lambda}$ of X, **finite** open sets $\{V_\lambda\}_{\lambda \in \Lambda}$ of $\mathbb{R}^n$, and **finite** homeomorphisms $\{\phi_\lambda : U_\lambda \to V_\lambda\}_{\lambda \in \Lambda}$ such that for any λ, ν with $U_\lambda \cap U_\nu \neq \emptyset$, $\phi_\lambda(U_\lambda \cap U_\nu)$ is definable and $\phi_\nu \circ \phi_\lambda^{-1} : \phi_\lambda(U_\lambda \cap U_\nu) \to \phi_\nu(U_\lambda \cap U_\nu)$ is a definable C^r diffeomorphism.

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- This pair $(U_\lambda, \phi_\lambda)$ of sets and homeomorphisms is called a *definable C^r coordinate system*.
- A definable C^r manifold is *affine* if it is definably C^r diffeomorphic to a definable C^r submanifold of some R^n.

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Theorem (Shiota (1986))

Any compact C^∞ manifold of positive dimension admits uncountably many nonaffine Nash manifold structures.

Examples of definable C^r manifolds

Example

(1) The n*-dimensional unit sphere* $S^n = \{(x_1, \dots, x_{n+1}) \in R^{n+1} | \sum_{i=1}^{n+1} x_i^2 = 1\}$ *is an* n*-dimensional definable* C^∞ *manifold.*
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- The above examples are affine Nash manifolds.

Imbeddings of definable C^∞ manifolds

Theorem ((2005))

For every o-minimal expansion $\mathcal{M}$ *of* $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$, *every definable* C^r *manifold is affine when* $0 \leq r < \infty$.

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Theorem (Berarducci and M. Otero (2001))

For every o-minimal expansion $\mathcal{N}$ *of a real closed field* $(R, +, \cdot, <)$, *every definably compact definable* C^r *manifold* X *is definably* C^r *diffeomorphic to a definable* C^r *submanifold of some* R^n *when* $0 \leq r < \infty$.

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For every exponential o-minimal expansion $\mathcal{M}$ of $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$, every definable C^∞ manifold is affine.

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- The assumption that $\mathcal{M}$ is exponential is necessary for Fischer's theorem.

Critical points and critical values

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  We say that a point $p \in X$ is a *critical point* of f if the differential of f at p is zero.
  If p is a critical point of f, then $f(p)$ is called a *critical value* of f.

Nondegenerate critical points

- Let p be a critical point of f and (U, u) a definable C^r coordinate system on X at p (i.e. U is a definable open subset of X containing p and u is a definable C^r diffeomorphism from U onto a definable open subset of $\mathbb{R}^n$ with $u(p) = 0$).

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  Direct computations show that the notion of nondegeniricity does not depend on the choice of a local coordinate system.
  Y. Peterzil and S. Starchenko (2007) introduced definable C^r Morse functions in an o-minimal expansion of the standard structure of a real closed field when $r \geq 2$.

Definable C^r topology

- Let $Def^r(R^n)$ denote the set of definable C^r functions on R^n.
  For each $f \in Def^r(R^n)$ and for each positive definable function $\epsilon : R^n \to R$, the ϵ-neighborhood $N(f;\epsilon)$ of f in $Def^r(R^n)$ is defined by
  $\{h \in Def^r(R^n) | |\partial^\alpha(h-f)| < \epsilon, \forall \alpha \in (\mathbb{N} \cup \{0\})^n, |\alpha| \le r\}$,
  where $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$,
  $|\alpha| = \alpha_1 + \cdots + \alpha_n, \partial^\alpha F = \frac{\partial^{|\alpha|} F}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$.
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  We call the topology defined by these ϵ-neighborhoods the *definable C^r topology*.
  Let X be a definable C^r submanifold of R^n. As in the same way, We can define the *definable C^r topology* of the set $Def^r(X)$ of definable C^r functions on X.

Our result

Theorem

Let X be a definably compact definable C^r manifold and $2 \leq r < \infty$. Then the set of definable Morse functions $Def^r_{Morse}(X)$ is open and dense in the set $Def^r(X)$ of definable C^r functions on X with respect to the definable C^2 topology.

- To prove Theorem, we need the following results.

Lemma (van den Dries (1998))

Let $A \subset R^n$ be a definable set which is the union of definable open subsets $U_1, \ldots, U_n$ of A. Then A is the union of definable open subsets $W_1, \ldots, W_n$ of A with $cl_A(W_i) \subset U_i$ for $i = 1, \ldots, n$, where $cl_A(W_i)$ denotes the closure of W_i in A.

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Theorem (Peterzil and Steinhorn (1999))

For a definable subset of R^n, it is definably compact if and only if it is closed and bounded.

Proof of Theorem

Theorem (Berarducci and M. Otero (2001))

Let $X \subset R^l$ *be a definable* C^r *manifold and* $0 \leq r < \infty$. *Given two disjoint definable sets* $F_0, F_1 \subset X$ *closed in* X, *there exists a definable* C^r *function* $\delta : X \to R$ *which is* 0 *exactly on* F_0, 1 *exactly on* F_2 *and* $0 \leq \delta \leq 1$.

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Theorem (Berarducci and M. Otero (2001))

Let $X \subset R^l$ be a definable C^r manifold and $0 \le r < \infty$. Given two disjoint definable sets $F_0, F_1 \subset X$ closed in X, there exists a definable C^r function $\delta : X \to R$ which is 0 exactly on F_0, 1 exactly on F_2 and $0 \le \delta \le 1$.

- The following result is a definable version of Sard's Theorem.

Theorem (Berarducci and M. Otero (2001))

Let $X_1 \subset R^s$ and $X_2 \subset R^t$ be definable C^r manifolds of dimension m and n, respectively. Let $f : X_1 \to X_2$ be a definable C^r map. Then the set of critical values of f has dimension less than n.

- By the above theorem, we have the following lemma.

Lemma

Let U be a definable open subset of R^m and $f : U \to R$ a definable C^r function. There exist $a_1, \dots, a_m \in R$ such that $F(x_1, \dots, x_m) = f(x_1, \dots, x_m) - (a_1x_1 + \cdots + a_mx_m)$ is a definable Morse function on U and $|a_1|, \dots, |a_m|$ are sufficiently small.

Proof of Theorem

- Let $\{\phi_i : U_i \to V_i\}_{i=1}^k$ be a definable C^r coordinate system of X. By the above results and since X is definably compact, shrinking $\{U_i\}_{i=1}^k$, if necessary, there exists a finite collection $\{K_i\}_{i=i}^k$ of definably compact subsets with $K_i \subset U_i$ such that $X = \cup_{i=1}^k K_i$. From now on we fix $\{U_i\}_{i=1}^k$ and $\{K_i\}_{i=1}^k$.

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  Let $f, g : X \to R$ be definable C^r functions and $\epsilon > 0$. We say that g is a *(C^2, ϵ) approximation of f* on a definably compact subset K of X if the following three inequalities hold for any point $p \in K$.

$$\begin{cases} |f(p) - g(p)| < \epsilon, \\ |\frac{\partial f}{\partial x_i}(p) - \frac{\partial g}{\partial x_i}(p)| < \epsilon, 1 \le i \le n, \\ |\frac{\partial^2 f}{\partial x_i \partial x_j}(p) - \frac{\partial^2 g}{\partial x_i \partial x_j}(p)| < \epsilon, 1 \le i, j \le n. \end{cases}$$

Proof of Theorem

Definition

Let $f : X \to R$ *be a definable* C^r *function and* $\epsilon > 0$. *A definable* C^r *function* $g : X \to R$ *is a* (C^2, ϵ) *approximation of* f *if* g *is a* (C^2, ϵ) *approximation of* f *on any* K_i.

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Proposition

Let C *be a definably compact subset of* X*,* $h : X \to R$ *a definable* C^r *function and* $\epsilon > 0$ *is sufficiently small. If there are no degenerate critical points of* h *in* C*, then for every definable* C^r *function* $h' : X \to R$ *which is a* (C^2, ϵ) *approximation of* h*,* C *does not contain a degenerate critical point of* h'*. In particular* $Def^r_{Morse}(X)$ *is open in* $Def^r(X)$ *with respect to the definable* C^2 *topology.*

- ***Proof of Proposition.***
  We consider in a definable C^r coordinate neighborhood $(U_l, (x_1, \dots, x_m))$. Let the Hessian of h with respect to $(U_l, (x_1, \dots, x_m))$ be $(\frac{\partial^2 h}{\partial x_i \partial x_j})$. Then h has no degenerate critical points in $C \cap K_l$ if and only if $|\frac{\partial h}{\partial x_1}| + \cdots + |\frac{\partial h}{\partial x_n}| + |\det(\frac{\partial^2 h}{\partial x_i \partial x_j})| > 0$ holds in $C \cap K_l$. If $\epsilon > 0$ is sufficiently small, then for any h' which is a (C^2, ϵ) approximation of h, $|\frac{\partial h'}{\partial x_1}| + \cdots + |\frac{\partial h'}{\partial x_n}| + |\det(\frac{\partial^2 h'}{\partial x_i \partial x_j})| > 0$ holds in $C \cap K_l$. Thus h' has no degenerate critical points in $C \cap K_l$. By a similar argument, h' has no degenerate critical points in $C = \cup_{i=1}^k C \cap K_l$.

- ***Proof of Our Theorem.***
  Proposition proves that $Def^r_{Morse}(X)$ is open in $Def^r(X)$.
  To prove density of $Def^r_{Morse}(X)$, we proceed by induction on l.
  Let $g : X \to R$ be a definable C^r function and $\epsilon > 0$. Assume that we have a definable C^r function $f_{l-1} : X \to R$ such that f_{l-1} has no degenerate critical points in $C_{l-1} := \cup_{i=1}^{l-1} K_i$ and it is a (C^2, δ_{l-1}) approximation of g, where $\delta_{l-1} > 0$ is sufficiently smaller than ϵ.
  We consider a definable C^r coordinate neighborhood $(U_l, (x_1, \dots, x_m))$. By Lemma, there exist $a_1, \dots, a_m \in R$ such that $f(x_1, \dots, x_m) - (a_1 x_1 + \cdots + a_m x_m)$ is a definable Morse function on U_l and $|a_1|, \dots, |a_m|$ are sufficiently small. By Theorem BO, we have a definable C^r function $h_l : X \to R$ such that h_l is identically 1 on some definable open neighborhood V_l of K_l in U_l, h_l is identically 0 outside of some definably compact set L_l with $V_l \subset L_l \subset U_l$ and $0 \le h_l \le 1$.

Proof of Theorem

- We define $f_l : X \to R, f_l = f_{l-1}(x_1, \dots, x_m) - (a_1x_1 + \cdots + a_mx_m)h_l(x_1, \dots, x_m)$ on U_l and $f_l = f_{l-1}(x_1, \dots, x_m)$ outside of L_l. By the definition of f_l, f_l is a definable C^r function on X.
  Calculating on U_l,
  $|f_{l-1}(p) - f_l(p)| = |a_1x_1 + \cdots + a_mx_m|h_l(p)$,
  $|\frac{\partial f_{l-1}}{\partial x_i}(p) - \frac{\partial f_l}{\partial x_i}(p)| = |a_ih_l(p) + (a_1x_1 + \cdots + a_mx_m)\frac{\partial h_l}{\partial x_i}(p)|, 1 \le i \le m$,
  $|\frac{\partial^2 f_{l-1}}{\partial x_i \partial x_j}(p) - \frac{\partial^2 f_l}{\partial x_i \partial x_j}(p)| = |a_i\frac{\partial h_l}{\partial x_j}(p) + a_j\frac{\partial h_l}{\partial x_i}(p) + (a_1x_1 + \cdots + a_mx_m)\frac{\partial^2 h_l}{\partial x_i \partial x_j}(p)|, 1 \le i, j \le m$,
  where $p = (x_1, \dots, x_m)$.
  By the construction of h_l and since X is definably compact, $|h_l|, |\frac{\partial h_l}{\partial x_i}|, |\frac{\partial^2 h_l}{\partial x_i \partial x_j}|$ are bounded. Thus f_l is a (C^2, δ'_l) approximation of f_{l-1} on K_l if $|a_1|, \dots, |a_m| > 0$ are sufficiently small.

Proof of Theorem

- We now consider on K_j when $j \neq l$. Since $f_{l-1} = f_l$ outside of L_l, we only have to evaluate them on $K_j \cap L_l$. Since $K_j \cap L_l \subset U_j \cap U_l$, they are evaluated by the Jacobian of $(U_j, (y_1, \dots, y_m))$ between $(U_l, (x_1, \dots, x_m))$. It is bounded on $K_j \cap L_l$ because $K_j \cap L_l$ is definably compact. Thus they are sufficiently small if $|a_1|, \dots, |a_m| > 0$ are sufficiently small. Hence f_l is a (C^2, δ_l) approximation of f_{l-1}. By Proposition, f_l has no degenerate critical points in C_{l-1}. By the construction of f_l, f_l has no degenerate critical points in K_l. Thus there are no degenerate critical points of f_l in $C_l := \cup_{i=1}^{l} K_i$. Therefore $f_k : X \to R$ is the required definable Morse function on X.