# O-minimal structures

Tomohiro kawakami

Wakayama University

2011.8.29

# Structures

- A structure $\mathcal{N} = (R, (S_i)_{i \in I}, (f_j)_{j \in J})$ consisting of
  a nonempty set $R$,
  relations $S_i \subset R^{m(i)}$ $(i \in I, m(i) \in \mathbb{N} \cup \{0\})$ and
  functions $f_j : R^{n(j)} \longrightarrow R$ $(j \in J, n(j) \in \mathbb{N} \cup \{0\})$
  If $n(j) = 0$, then we identify $f_j$ with its unique vale in $R$, and call $f_j$ a constant.
  $R$ is called the universe of $\mathcal{N}$ or the underlying set of $\mathcal{N}$

# Structures

- A structure $\mathcal{N} = (R, (S_i)_{i \in I}, (f_j)_{j \in J})$ consisting of
  a nonempty set $R$,
  relations $S_i \subset R^{m(i)}$ $(i \in I, m(i) \in \mathbb{N} \cup \{0\})$ and
  functions $f_j : R^{n(j)} \to R$ $(j \in J, n(j) \in \mathbb{N} \cup \{0\})$
  If $n(j) = 0$, then we identify $f_j$ with its unique vale in $R$, and call $f_j$ a constant.
  $R$ is called the universe of $\mathcal{N}$ or the underlying set of $\mathcal{N}$
  We say that $f$ (resp. $R$) is an $m$-place function (resp. an $m$-place relation) if $f : R^m \to R$ is a function (resp. $R \subset R^m$ is a relation).

# Terms

# Terms

A term is a finite string of symbols obtained by repeated applications of the following two rules:

1. Variables are terms.
2. If $f$ is an $m$-place function of $\mathcal{N}$ and $t_1, \ldots, t_m$ are terms, then the concatenated string $f(t_1, \ldots, t_m)$ is a term.

Note that if $m = 0$, then the second rule says that constants ($0$-place function) are terms.

# Formulas

# Formulas

A formula is a finite string of symbols $s_1 \ldots s_k$, where each $s_i$ is either a variable, a function symbol, a relation symbol, one of the logical symbols $=, \neg, \vee, \wedge, \exists, \forall$, one of the brackets (, ), or comma ,. Arbitrary formulas are generated inductively by the following three rules:

1. For any two terms $t_1$ and $t_2$, $t_1 = t_2$ is a formula.
2. If $R$ is an $m$-place relation and $t_1, \ldots, t_m$ are terms, then $R(t_1, \ldots, t_m)$ is a formula.
3. If $\phi$ and $\psi$ are formulas, then the negation $\neg\phi$, the disjunction $\phi \vee \psi$, and the conjunction $\phi \wedge \psi$ are formulas. If $\phi$ is a formula and $v$ is a variable, then $(\exists v)\phi$ and $(\forall v)\phi$ are formulas.

# Dense linear order without endpoints

# Dense linear order without endpoints

Let $\mathcal{N} = (R, <, \cdots)$ be an ordered structure.
The order $<$ is linear if for any $x, y \in R$, exactly one of $x < y, x = y, x > y$ holds.
We say that $<$ is dense if for all $x, y \in R$ with $x < y$, there exists $z \in R$ with $x < z < y$, and say that $<$ has no endpoints if for any $x \in R$, there exist $y, z \in R$ such that $y < x < z$.

# Dense linear order without endpoints

Let $\mathcal{N} = (R, <, \cdots)$ be an ordered structure.
The order $<$ is linear if for any $x, y \in R$, exactly one of $x < y, x = y, x > y$ holds.
We say that $<$ is dense if for all $x, y \in R$ with $x < y$, there exists $z \in R$ with $x < z < y$, and say that $<$ has no endpoints if for any $x \in R$, there exist $y, z \in R$ such that $y < x < z$.
For convenience, we add two endpoint $-\infty$ and $\infty$, with $-\infty < x < \infty$ for all $x \in R$.
An open interval $(a, b)$ means $\{x \in R | a < x < b\}$ with $-\infty \leq a < b \leq \infty$.

# Dense linear order without endpoints

Let $\mathcal{N} = (R, <, \cdots)$ be an ordered structure.
The order $<$ is linear if for any $x, y \in R$, exactly one of $x < y, x = y, x > y$ holds.
We say that $<$ is dense if for all $x, y \in R$ with $x < y$, there exists $z \in R$ with $x < z < y$, and say that $<$ has no endpoints if for any $x \in R$, there exist $y, z \in R$ such that $y < x < z$.
For convenience, we add two endpoint $-\infty$ and $\infty$, with $-\infty < x < \infty$ for all $x \in R$.
An open interval $(a, b)$ means $\{x \in R | a < x < b\}$ with $-\infty \leq a < b \leq \infty$.
From now on, we only consider a dense linearly ordered structure $\mathcal{N} = (R, <, \cdots)$ without endpoints.

# O-minimal structures (Order minimal structures)

We call $\mathcal{N}$ o-minimal (order minimal) if every definable subset of $\boldsymbol{R}$ is a finite union of points and open intervals.

# O-minimal structures (Order minimal structures)

We call $\mathcal{N}$ o-minimal (order minimal) if every definable subset of $\boldsymbol{R}$ is a finite union of points and open intervals.

From now on, we only consider an o-minimal structure $\mathcal{N} = (\boldsymbol{R}, <, \cdots)$.

# Examples of o-minimal structures

- O-minimal structures are a class of ordered structures generalizing interesting classical structures such as:

# Examples of o-minimal structures

- O-minimal structures are a class of ordered structures generalizing interesting classical structures such as:
  (1) The field $\mathbb{R}$ of real numbers.

## Examples of o-minimal structures

- O-minimal structures are a class of ordered structures generalizing interesting classical structures such as:
  (1) The field $\mathbb{R}$ of real numbers.
  (2) $\mathbf{R}_{an} := (\mathbb{R}, +, \cdot, <, (f))$, where $f$ ranges over all restricted analytic functions, namely all functions $\mathbb{R}^n \longrightarrow \mathbb{R}, n \in \mathbb{N}$ that vanish identically outside $[-1, 1]^n$ and whose restrictions to $[-1, 1]^n$ are analytic.

- O-minimal structures are a class of ordered structures generalizing interesting classical structures such as:
  (1) The field $\mathbb{R}$ of real numbers.
  (2) $\mathbf{R}_{an} := (\mathbb{R}, +, \cdot, <, (f))$, where $f$ ranges over all restricted analytic functions, namely all functions $\mathbb{R}^n \longrightarrow \mathbb{R}, n \in \mathbb{N}$ that vanish identically outside $[-1, 1]^n$ and whose restrictions to $[-1, 1]^n$ are analytic.
  (3) $\mathbf{R}_{an}^S := (\mathbb{R}, +, \cdot, <, (f), (x^r)_{r \in S})$, where $S$ is a subset of $\mathbb{R}$, $f$ ranges over all restricted analytic functions as in (2), and the function $x^r : \mathbb{R} \longrightarrow \mathbb{R}$ is given by

$$a \mapsto \begin{cases} a^r, & a > 0 \\ 0, & a \leq 0 \end{cases} .$$

# Examples of o-minimal structures

- O-minimal structures are a class of ordered structures generalizing interesting classical structures such as:
  (1) The field $\mathbb{R}$ of real numbers.
  (2) $\mathbf{R}_{an} := (\mathbb{R}, +, \cdot, <, (f))$, where $f$ ranges over all restricted analytic functions, namely all functions $\mathbb{R}^n \longrightarrow \mathbb{R}, n \in \mathbb{N}$ that vanish identically outside $[-1, 1]^n$ and whose restrictions to $[-1, 1]^n$ are analytic.
  (3) $\mathbf{R}_{an}^S := (\mathbb{R}, +, \cdot, <, (f), (x^r)_{r \in S})$, where $S$ is a subset of $\mathbb{R}$, $f$ ranges over all restricted analytic functions as in (2), and the function $x^r : \mathbb{R} \longrightarrow \mathbb{R}$ is given by

$$a \mapsto \begin{cases} a^r, & a > 0 \\ 0, & a \le 0 \end{cases} .$$

  (4) $\mathbf{R}_{exp} := (\mathbb{R}, +, \cdot, <, exp)$, where $exp : \mathbb{R} \longrightarrow \mathbb{R}$ denotes the exponential function $x \mapsto e^x$.

# Examples of o-minimal structures

- O-minimal structures are a class of ordered structures generalizing interesting classical structures such as:
  (1) The field $\mathbb{R}$ of real numbers.
  (2) $\mathbf{R}_{an} := (\mathbb{R}, +, \cdot, <, (f))$, where $f$ ranges over all restricted analytic functions, namely all functions $\mathbb{R}^n \longrightarrow \mathbb{R}, n \in \mathbb{N}$ that vanish identically outside $[-1, 1]^n$ and whose restrictions to $[-1, 1]^n$ are analytic.
  (3) $\mathbf{R}_{an}^S := (\mathbb{R}, +, \cdot, <, (f), (x^r)_{r \in S})$, where $S$ is a subset of $\mathbb{R}$, $f$ ranges over all restricted analytic functions as in (2), and the function $x^r : \mathbb{R} \longrightarrow \mathbb{R}$ is given by

$$a \mapsto \begin{cases} a^r, & a > 0 \\ 0, & a \leq 0 \end{cases}.$$

  (4) $\mathbf{R}_{exp} := (\mathbb{R}, +, \cdot, <, exp)$, where $exp : \mathbb{R} \longrightarrow \mathbb{R}$ denotes the exponential function $x \mapsto e^x$.
  (5) $\mathbf{R}_{an,exp} := (\mathbb{R}, +, \cdot, <, (f), exp)$, where $(f)$ and $exp$ denote as above.

We equip $\boldsymbol{R}$ with the interval topology (the intervals form a base), and each product $\boldsymbol{R}^n$ with the corresponding product topology.
Note that $\boldsymbol{R}^n$ is a Hausdorff space with this topology.

# Monotonicity theorem

# Monotonicity theorem

Let $\mathcal{N} = (R, <, \dots)$ be an o-minimal structure and $X \subset R^n, Y \subset R^m$ definable sets. We say that a map $f : X \to Y$ is a definable map if the graph $\{(x, f(x)) \in X \times Y | x \in X\} \subset R^n \times R^m$ is a definable set.

Let $\mathcal{N} = (R, <, \dots)$ be an o-minimal structure and $X \subset R^n, Y \subset R^m$ definable sets. We say that a map $f : X \to Y$ is a definable map if the graph $\{(x, f(x)) \in X \times Y | x \in X\} \subset R^n \times R^m$ is a definable set.

**Theorem (Monotonicity theorem)**

*Let $f : (a, b) \to R$ be a definable function on the interval $(a, b)$. Then there exist points $a = a_0 < a_1 < \cdots < a_k < a_{k+1} = b$ in $(a, b)$ such that on each subinterval $(a_j, a_{j+1})$, the restriction $f|(a_j, a_{j+1})$ is either constant, or strictly monotone and continuous.*

# Cell decomposition theorem

Let $\mathcal{N} = (R, <, \dots)$ be an o-minimal structure.
For each definable set $X$ in $R^n$, we put

$$C(X) = \{f : X \to R | f \text{ is definable and continuous } \},$$

$$C_\infty(X) = C(X) \cup \{+\infty, -\infty\},$$

where we regard $+\infty$ and $-\infty$ as constant functions on $X$.

# Cell decomposition theorem

Let $\mathcal{N} = (R, <, \dots)$ be an o-minimal structure.
For each definable set $X$ in $R^n$, we put

$$C(X) = \{f : X \to R | f \text{ is definable and continuous } \},$$

$$C_\infty(X) = C(X) \cup \{+\infty, -\infty\},$$

where we regard $+\infty$ and $-\infty$ as constant functions on $X$.
For $f, g \in C_\infty(X)$, we write $f < g$ if $f(x) < g(x)$ for all $x \in X$, and in this case we put

$$(f, g)_X = \{(x, r) \in X \times R | f(x) < r < g(x)\}.$$

$(f, g)_X$ is a definable subset of $R^{n+1}$.

# Cell decomposition theorem

# Cell decomposition theorem

Let $(i_1, \ldots, i_n)$ be a sequence of zeros and ones of length $n$. An $(i_1, \ldots, i_n)$ cell is a definable subset of $R^n$ obtained by induction on $n$ as follows:

1. a $(0)$ cell is a one-element set $\{r\} \subset R$ (a point), a $(1)$ cell is an interval $(a, b) \subset R$.
2. suppose $(i_1, \ldots, i_n)$ cells are already defined. Then an $(i_1, \ldots, i_n, 0)$ cell is the graph $\Gamma(f)$ of a function $f \in C(X)$, where $X$ is an $(i_1, \ldots, i_n)$ cell. As $(i_1, \ldots, i_n, 1)$ cell is a set $(f, g)_X$, where $X$ is an $(i_1, \ldots, i_n)$ cell and $f, g \in C_\infty(X), f < g$.

# Cell decomposition theorem

Let $(i_1, \ldots, i_n)$ be a sequence of zeros and ones of length $n$. An $(i_1, \ldots, i_n)$ cell is a definable subset of $R^n$ obtained by induction on $n$ as follows:

1. a $(0)$ cell is a one-element set $\{r\} \subset R$ (a point), a $(1)$ cell is an interval $(a, b) \subset R$.
2. suppose $(i_1, \ldots, i_n)$ cells are already defined. Then an $(i_1, \ldots, i_n, 0)$ cell is the graph $\Gamma(f)$ of a function $f \in C(X)$, where $X$ is an $(i_1, \ldots, i_n)$ cell. As $(i_1, \ldots, i_n, 1)$ cell is a set $(f, g)_X$, where $X$ is an $(i_1, \ldots, i_n)$ cell and $f, g \in C_\infty(X), f < g$.

A cell in $R^n$ is an $(i_1, \ldots, i_n)$ cell, for some sequence $(i_1, \ldots, i_n)$. The $(1, \ldots, 1)$ cells are exactly the cells which are open in $R^n$.

# Cell decomposition theorem

# Cell decomposition theorem

## Definition

*A decomposition of $R^n$ is a special kind of partition of $R^n$ into finitely many cells. The definition is by induction on $n$.*

1. *A decomposition of $R$ is a collection*

$$\{(-\infty, a_1), (a_1, a_2), \ldots, (a_k, \infty), \{a_1\}, \ldots, \{a_k\}\},$$

   *where $a_1 < \cdots < a_k$ are points in $R$.*

2. *A a finite partition of $R^{n+1}$ into cells $A$ such that the set of projections $\pi(A)$ is a decomposition of $R^n$, where $\pi : R^{n+1} \to R^n, \pi(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n)$.*

3. *A decomposition $\mathcal{D}$ of $R^n$ is said to partition a set $S \subset R^n$ if each cell in $\mathcal{D}$ is either part of $S$ or disjoint from $S$.*

# Cell decomposition theorem

# Cell decomposition theorem

**Theorem (Cell decomposition theorem)**

1. *Given any definable sets* $A_1, \ldots, A_k \subset R^n$*, there exists a decomposition of* $R^n$ *partitioning each of* $A_1, \ldots, A_k$*.*
2. *For each definable function* $f : A \longrightarrow R$, $A \subset R^n$*, there exists a decomposition* $\mathcal{D}$ *of* $R^n$ *partitioning* $A$ *such that the restriction* $f|B : B \longrightarrow R$ *to each cell* $B \in \mathcal{D}$ *with* $B \subset A$ *is continuous.*

# Cell decomposition theorem

**Theorem (Cell decomposition theorem)**

1. *Given any definable sets* $A_1, \dots, A_k \subset R^n$, *there exists a decomposition of* $R^n$ *partitioning each of* $A_1, \dots, A_k$.
2. *For each definable function* $f : A \to R$, $A \subset R^n$, *there exists a decomposition* $\mathcal{D}$ *of* $R^n$ *partitioning* $A$ *such that the restriction* $f|B : B \to R$ *to each cell* $B \in \mathcal{D}$ *with* $B \subset A$ *is continuous.*

A set $Y \subset R^{n+1}$ is finite over $R^n$ if for each $x \in R^n$, the fiber $Y_x = \{r \in R | (x, r) \in Y\}$ is finite. We call $Y$ uniformly finite over $R^n$ if there exists $N \in \mathbb{N}$ such that $|Y_x| \leq N$ for all $x \in R^n$.

**Theorem (Uniform finiteness)**

*Suppose that a definable subset* $A$ *of* $R^{n+1}$ *is finite over* $R^n$. *Then* $Y$ *is uniformly finite.*

# $C^r$ Cell decomposition theorem

# $C^r$ Cell decomposition theorem

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of a real closed field.

Let $X \subset R^n$ be a definable open set and $r$ a positive integer. A definable map $f : X \to R^n$ is a definable $C^r$ map if $f$ is of class $C^r$.

A definable map $f : A \to R^n$, where $A \subset R^m$ is not necessarily open, is a definable $C^r$ map if there exist a definable open set $U \subset R^m$ containing $A$ and a definable $C^r$ map $F : U \to R^n$ such that $f = F|A$.

# $C^r$ Cell decomposition theorem

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of a real closed field.

Let $X \subset R^n$ be a definable open set and $r$ a positive integer. A definable map $f : X \to R^n$ is a definable $C^r$ map if $f$ is of class $C^r$.

A definable map $f : A \to R^n$, where $A \subset R^m$ is not necessarily open, is a definable $C^r$ map if there exist a definable open set $U \subset R^m$ containing $A$ and a definable $C^r$ map $F : U \to R^n$ such that $f = F|A$.

we put

$$C^r(X) = \{f : X \to R | f \text{ is definable and of class } C^r\},$$

$$C^r_\infty(X) = C^r(X) \cup \{+\infty, -\infty\},$$

where we regard $+\infty$ and $-\infty$ as constant functions on $X$.

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of a real closed field.
Let $X \subset R^n$ be a definable open set and $r$ a positive integer. A definable map $f : X \to R^n$ is a definable $C^r$ map if $f$ is of class $C^r$.
A definable map $f : A \to R^n$, where $A \subset R^m$ is not necessarily open, is a definable $C^r$ map if there exist a definable open set $U \subset R^m$ containing $A$ and a definable $C^r$ map $F : U \to R^n$ such that $f = F|A$.
we put

$$C^r(X) = \{f : X \to R | f \text{ is definable and of class } C^r\},$$

$$C^r_\infty(X) = C^r(X) \cup \{+\infty, -\infty\},$$

where we regard $+\infty$ and $-\infty$ as constant functions on $X$.
For $f, g \in C^r_\infty(X)$, we write $f < g$ if $f(x) < g(x)$ for all $x \in X$, and in this case we put

$$(f, g)_X = \{(x, r) \in X \times R | f(x) < r < g(x)\}.$$

Let $(i_1, \ldots, i_n)$ be a sequence of zeros and ones of length $n$. An $(i_1, \ldots, i_n)$ $C^r$ cell is a definable subset of $R^n$ obtained by induction on $n$ as follows:

1. a $(0)$ $C^r$ cell is a one-element set $\{r\} \subset R$ (a point), a $(1)$ $C^r$ cell is an interval $(a, b) \subset R$.
2. suppose $(i_1, \ldots, i_n)$ cells are already defined. Then an $(i_1, \ldots, i_n, 0)$ $C^r$ cell is the graph $\Gamma(f)$ of a function $f \in C^r(X)$, where $X$ is an $(i_1, \ldots, i_n)$ $C^r$ cell. As $(i_1, \ldots, i_n, 1)$ $C^r$ cell is a set $(f, g)_X$, where $X$ is an $(i_1, \ldots, i_n)$ $C^r$ cell and $f, g \in C^r_\infty(X), f < g$.

# $C^r$ Cell decomposition theorem

## Definition

*A decomposition of $\boldsymbol{R}^n$ is a special kind of partition of $\boldsymbol{R}^n$ into finitely many $C^r$ cells. The definition is by induction on $n$.*

1. *A decomposition of $\boldsymbol{R}$ is a collection*

$$\{(-\infty, a_1), (a_1, a_2), \dots, (a_k, \infty), \{a_1\}, \dots, \{a_k\}\},$$

   *where $a_1 < \cdots < a_k$ are points in $\boldsymbol{R}$.*

2. *A a finite partition of $\boldsymbol{R}^{n+1}$ into $C^r$ cells $A$ such that the set of projections $\pi(A)$ is a decomposition of $\boldsymbol{R}^n$, where $\pi : \boldsymbol{R}^{n+1} \to \boldsymbol{R}^n, \pi(x_1, \dots, x_n, x_{n+1}) = (x_1, \dots, x_n)$.*
3. *A decomposition $\mathcal{D}$ of $\boldsymbol{R}^n$ is said to partition a set $S \subset \boldsymbol{R}^n$ if each $C^r$ cell in $\mathcal{D}$ is either part of $S$ or disjoint from $S$.*

# $C^r$ Cell decomposition theorem

**Theorem ($C^r$ Cell decomposition theorem)**

1. *Given any definable sets* $A_1, \dots, A_k \subset R^n$*, there exists a decomposition of* $R^n$ *partitioning each of* $A_1, \dots, A_k$.
2. *For each definable function* $f : A \to R, A \subset R^n$*, there exists a decomposition* $\mathcal{D}$ *of* $R^n$ *partitioning* $A$ *such that the restriction* $f|B : B \to R$ *to each* $C^r$ *cell* $B \in \mathcal{D}$ *with* $B \subset A$ *is of class* $C^r$.

# Triangulation

# Triangulation

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of a real closed field.
An affine subspace of $R^n$ of dimension $d$ is a translate $L + a$ of a linear subspace $L$ of $R^n$ of dimension $d$.
A tuple $a_0, \dots, a_k$ of points in $R^n$ is affine independent if the smallest affine subspace containing $a_0, \dots, a_k$ has dimension $k$.

# Triangulation

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of a real closed field.
An affine subspace of $R^n$ of dimension $d$ is a translate $L + a$ of a linear subspace $L$ of $R^n$ of dimension $d$.
A tuple $a_0, \dots, a_k$ of points in $R^n$ is affine independent if the smallest affine subspace containing $a_0, \dots, a_k$ has dimension $k$.
An affine independent tuple of points $a_0, \dots a_k$ of $R^n$, we call $(a_0, \dots, a_k) = \{\sum_{i=1}^k t_i a_i |$ all $t_i > 0, \sum_{i=1}^k t_i = 1\}$ a $k$-simplex.

# Triangulation

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of a real closed field.
An affine subspace of $R^n$ of dimension $d$ is a translate $L + a$ of a linear subspace $L$ of $R^n$ of dimension $d$.
A tuple $a_0, \dots, a_k$ of points in $R^n$ is affine independent if the smallest affine subspace containing $a_0, \dots, a_k$ has dimension $k$.
An affine independent tuple of points $a_0, \dots a_k$ of $R^n$, we call $(a_0, \dots, a_k) = \{\sum_{i=1}^k t_i a_i |$ all $t_i > 0, \sum_{i=1}^k t_i = 1\}$ a $k$-simplex.
A complex $K$ in $R^n$ is a finite collection of simplexes in $R^n$ such that for all $\sigma_1, \sigma_2 \in K$, either $\overline{\sigma_1} \cap \overline{\sigma_2} = \emptyset$, or $\overline{\sigma_1} \cap \overline{\sigma_2} = \overline{\tau}$ for some common face $\tau$ of $\sigma_1$ and $\sigma_2$. Here $\overline{\sigma_1}$ (resp. $\overline{\sigma_2}$, $\overline{\tau}$) denotes the closure of $\sigma_1$ (resp. $\sigma_2$, $\tau$) in $R^n$. Notice that $\tau$ is not required to belong to $K$. A complex is called closed if it contains all its faces of each simplex. Note that a complex $K$ in $R^n$ is closed if and only if $|K|$ is closed in $R^n$.

# Triangulation

# Triangulation

## Definition

*Let $A \subset R^n$ be a definable set. A definable triangulation in $R^n$ of $A$ is a pair $(\psi, K)$ consisting of a complex $K$ in $R^n$ and a definable homeomorphism $\psi\colon A \longrightarrow |K|$. The triangulation is said to be compatible with a definable subset $B \subset A$ if $B$ is a union of some elements of $\psi^{-1}(K)$.*

**Definition**

*Let $A \subset R^n$ be a definable set. A definable triangulation in $R^n$ of $A$ is a pair $(\psi, K)$ consisting of a complex $K$ in $R^n$ and a definable homeomorphism $\psi\colon A \longrightarrow |K|$. The triangulation is said to be compatible with a definable subset $B \subset A$ if $B$ is a union of some elements of $\psi^{-1}(K)$.*

**Theorem (Triangulation theorem)**

*Let $S \subset R^n$ be a definable set and $S_1, \dots S_k$ definable subsets of $S$. Then $S$ has a triangulation in $R^n$ which is compatible with $S_1, \dots, S_k$.*

# Piecewise trivialization theorem

# Piecewise trivialization theorem

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of a real closed field.

## Definition

*Let $A \subset R^m$, $S \subset R^n$ be definable sets, and let $f : S \to A$ be a definable continuous map. We say that $f$ is definably trivial if there exist a definable set $F \subset R^N$ for some $N \in \mathbb{N}$, and a definable continuous map $h : S \to F$ such that $(f, h) : S \to A \times F$ is a definable homeomorphism. In this case, each fiber $f^{-1}(a)$ of $f$ over $a$ is definably homeomorphic to $F$. For a definable subset $B \subset A$, we call $f$ definably trivial over $B$ if the restriction $f|_{f^{-1}(B)} : f^{-1}(B) \to B$ is definably trivial.*

# Piecewise trivialization theorem

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of a real closed field.

## Definition

*Let $A \subset R^m$, $S \subset R^n$ be definable sets, and let $f : S \to A$ be a definable continuous map. We say that $f$ is definably trivial if there exist a definable set $F \subset R^N$ for some $N \in \mathbb{N}$, and a definable continuous map $h : S \to F$ such that $(f, h) : S \to A \times F$ is a definable homeomorphism. In this case, each fiber $f^{-1}(a)$ of $f$ over $a$ is definably homeomorphic to $F$. For a definable subset $B \subset A$, we call $f$ definably trivial over $B$ if the restriction $f|_{f^{-1}(B)} : f^{-1}(B) \to B$ is definably trivial.*

## Theorem (Piecewise trivialization theorem)

*Let $f : S \to A$ be a definable continuous map. Then there exists a finite partition $A = A_1 \cup \cdots \cup A_m$ of $A$ into definable sets $A_i$ such that $f$ is definably trivial over each $A_i$.*

# Definable quotient

# Definable quotient

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of a real closed field.

## Definition

*Let $E \subset X \times X$ be a definable equivalence relation on a definable set $X \subset R^n$. A definably proper quotient of $X$ by $E$ is a pair $(p, Y)$ consisting of a definable set $Y \subset R^m$ and a definable continuous surjective map $p : X \to Y$ such that*

1. $E = E_p$, *that is* $(x_1, x_2) \in E \Leftrightarrow p(x_1) = p(x_2)$ *for all* $x_1, x_2 \in X$.
2. *For each definable set* $K \subset Y$, *if* $p^{-1}(K)$ *is closed and bounded in* $R^n$ , *then* $K$ *is closed and bounded in* $R^m$.

# Definable quotient

# Definable quotient

## Definition

*Let $E$ be a definable equivalence relation on a definable set $X$ and $pr_1 : X \times X \to X, pr_2 : X \times X \to X$ the restrictions of the two projections $X \times X \to X$. We call $E$ definably proper over $X$ if $pr_1$ is a definably proper map.*

# Definable quotient

# Definable quotient

**Theorem**

*Suppose the definable equivalence relation $E$ on the definable set $X$ is definably proper over $X$. Then $X/E$ exists as a definably proper quotient of $X$.*
*Namely $X/E$ is a definable set and the projection $p : X \to X/E$ is a definably proper definable surjective continuous map.*

# Definable quotient

**Theorem**

*Suppose the definable equivalence relation $E$ on the definable set $X$ is definably proper over $X$. Then $X/E$ exists as a definably proper quotient of $X$.*
*Namely $X/E$ is a definable set and the projection $p : X \to X/E$ is a definably proper definable surjective continuous map.*

**Theorem**

*If $X \subset R^n$ is closed and bounded and $E \subset X \times X$ is a closed definable equivalence relation, then $X/E$ exists as a definably proper quotient of $X$.*
*Namely $X/E$ is a definable set and the projection $p : X \to X/E$ is a definably proper definable surjective continuous map.*

# Definable quotient

# Definable quotient

A definable subset $X \subset R^n$ is definably compact if for any definable map $f : (a, b) \to X$, there exist the limits $\lim_{x \to a+0} f(x), \lim_{x \to b-0} f(x)$ in $X$.

**Theorem (Peterzil and Steinhorn 1999)**

*For a definable subset of $R^n$, it is definably compact if and only if it is closed and bounded.*

A definable set $G \subset R^n$ is a definable group if $G$ is a group and the group operations $G \times G \to G, G \to G$ are definable and continuous.

**Corollary**

*If $G$ is a definably compact group and $G$ acts a definable set $X$ definably and continuously, then the orbit space $X/G$ is a definable set and the orbit map $p : X \to X/G$ is a definably proper definable surjective continuous map.*